Kangweon-Kyungki Math. Jour. **9** (2001), No. 2, pp. 75–82

C-TRANSFORMATIONS ON OPEN 3-CELLS

Joo Sung Lee

Abstract. In this paper, we show the existence of open subsets U of S^3 which admit a C-transformation onto the interior of B^3. Let U be an open set which is homeomorphic to the interior of B^3. Then, we prove that if U has a pseudo general polyhedral prime end structure then U admits a C-transformation.

1. Introduction

The existence of a C-transformation onto the interior of some compact manifold M with nonempty boundary is one of the interesting unsolved problems in geometric topology. This work continues the earlier work of the author and B. Brechner [B-L], where they develop a 3-dimensional prime end theory, define a C-transformation, and prove the following: If U is an open subset of S^3 which admits a C-transformation ϕ onto the interior of a compact 3-manifold M with nonempty boundary, then (1) ϕ is uniformly continuous on the *collection* of all crosscuts of U and (2) The Induced Homeomorphism Theorem holds; that is, if h is a homeomorphism of $Cl(U)$ onto itself, then, $\phi h \phi^{-1}$ extends to a homeomorphism of all of M^3 onto itself.

Here, we give a *partial* answer to a problem of the earlier paper[B-L], by characterizing those open 3-cells of E^3, which admits C-transformation. Recall that L. Husch[Hu], C.T.C. Wall[W] and C. Edwards[E] provided sufficient conditions for an open 3-cell U to be homeomorphic to the interior of B^3, where B^3 is the closed 3-ball. Therefore, we will assume that our open set U is such an open set in this paper.

Received January 5, 2000.
2000 Mathematics Subject Classification: 54H15, 54H20.
Key words and phrases: prime ends, C-transformation, open 3-cell.
This work was supported by the research program of Dongguk University.

In this paper we will show that the existence of a *pseudo general polyhedral* prime end structure of U will guarantee the existence of a C-transformation from U onto the interior of B^3 and therefore we conclude that if U has a pseudo general polyhedral prime end structure then U admits the induced homeomorphism theorem[B-L].

2. Definitions

In this section we will briefly review three dimensional prime end theory. *Prime end theory* is essentially a compactification theory for simply connected, bounded domains, U, in E^2, or simply connected domains in S^2 with nondegenerate complement. The planar case was originally due to Caratheodory [C], and was later generalized to the sphere by Ursell and Young [U-Y]. The author and B. Brechner developed a *simple* three dimensional prime end theory for certain open subsets of Euclidean three space [B-L]. One of the main purposes in this theory is to focus an induced homeomorphism theorem which, we believe, have many applications.

Let U be a bounded connected open subset in E^3 with a finitely generated homology and a finitely generated fundamental group. We state basic definitions for prime end theory, a crosscut, a chain of crosscuts and a prime end. We refer to the reader [B-L] for the details including definitions and interesting examples.

1. A *crosscut* is an open 2-cell D in U such that
   (1) D separates U into exactly two complementary domains,
   (2) $Cl(D)$ is a 2-cell, and
   (3) $Cl(D) \cap Bd(U) = Bd(D)$.
2. A *chain of crosscuts* in U is a sequence $\{D_i\}_{i=1}^{\infty}$ of crosscuts such that
   (1) D_{i+1} separates D_i from $\{D_{i+j}\}_{j=2}^{\infty}$;
   (2) $Cl(D_i) \cap Cl(D_j) = \phi$, for $i \neq j$; and
   (3) $lim_{i\to\infty}(diam(D_i)) = 0$.
3. Two chains of crosscuts, $\{Q_i\}$ and $\{R_i\}$, are *equivalent* iff
   (1) For each Q_i, there exists $j > i$ such that Q_{i+1} separates Q_i from $Q_j \cup R_j$;
   (2) For each R_i, there exists $j > i$ such that R_{i+1} separates R_i from $R_j \cup Q_j$.

That is, two subsequences can be alternated or "interspersed" to form a new, equivalent chain of crosscuts.

4. A *prime end* of U is an equivalence class of chains of crosscuts of U.

Definition 2.1. A bounded domain U has a *prime end structure* iff there exists a finite number of prime ends, $\{E_i\}_{i=1}^n$, of U, and a finite number of crosscuts, $\{Q_i\}_{i=1}^n$, with Q_i a crosscut of some chain representing E_i, such that

1. $diam(Q_i) < \epsilon$,
2. If U_i denotes the corresponding domain for Q_i, then $Bd(U) \bigcup \cup_{i=1}^n U_i$ is an $\epsilon-$neighborhood of $Bd(U)$ in $Cl(U)$, and
3. If B_i denotes the corresponding boundary compactum of Q_i, then $\cup_{i=1}^n (B_i - S_i) = Bd(U)$ where $S_i = Bd(Q_i)$.

Definition 2.2. Let U be a bounded domain which has a prime end structure. If the closure of each crosscut Q_i is a polyhedral disk in the definition of prime end structure, then we say that U has a polyhedral prime end structure. In particular if $Bd(Q_i) \cap Bd(Q_j)$ is even number of points then we say that U has a pseudo general polyhedral prime end structure.

Remark. We do not give any conditions of intersection of interior of crosscuts in the above. In fact, it may happen that the intersection of interior of crosscuts is finite union of disks. But we can modify our crosscuts so that the intersection is finite union of disjoint circles [Lemma 3.2].

Definition 2.3. An *onto* homeomorphism $\phi : U \to Int(M^3)$, where M^3 is a compact 3-manifold, is called a *C-transformation* or a *C-map* iff

(1) The image of every chain of crosscuts of U is a chain of crosscuts of $Int(M^3)$. In particular, the image of each crosscut of U is a crosscut of $Int(M^3)$.
(2) On each crosscut Q, ϕ extends to a homeomorphism from $Cl(Q)$ onto $Cl(\phi(Q))$. (However, ϕ does not necessarily extend to a homeomorphism from the union of the closures of all the crosscuts of U to the union of the closures of their images in $\phi(Q)$.)
(3) For each crosscut Q_i of a prime end of U, let U_i be its corresponding domain. Let $(Q_i{}', U_i{}')$ be the image of (Q_i, U_i) under ϕ. We consider the following open sets on $Bd(M^3)$: $Int[Cl(U_i') \cap Bd(M^3)]$.

We require that the collection of all such open disks on $Bd(M^3)$ form a basis for the topology of $Bd(M^3)$.

Notation. We will use notation Q_i, S_i, U_i and B_i, respectively, where Q_i is a crosscut, $S_i = Bd(Q_i)$, U_i is the corresponding complementary domain of Q_i and $B_i = Cl(U_i) \cap Bd(U)$.

3. Main Theorems

In this section, we will prove the main theorem: If U has a pseudo general polyhedral prime end structure, then there exists a C-map $h : U \to Int(B^3)$. We will use *push and pull back* technique which is one of the basic technique in geometric topology. We refer to the reader [B1,B2] for the details.

We remark that there are some theorems which provide sufficient conditions for an open 3-manifold to be homeomorphic to E^3 ([E],[Hu],[W]). Therefore we give the following standing hypothesis in this section.

Standing Hypothesis for This Section: U is an open set which is homeomorphic to the interior of B^3.

LEMMA 3.1. *Let $\epsilon > 0$ and $\{Q_i\}_{i=1}^n$ a finite number of crosscuts corresponding to ϵ in the definition of pseudo polyhedral prime end structure. Suppose that Q_i and Q_j be crosscuts such that $Bd(Q_i) \cap Bd(Q_j) = \emptyset$. Then there exists polyhedral crosscuts $Q_i^{'}$ and $Q_j^{'}$ such that $Q_i^{'} \cap Q_j^{'} = \emptyset$, $Bd(Q_i) = Bd(Q_i^{'})$ and $Bd(Q_j) = Bd(Q_j^{'})$.*

Proof. We may assume that Q_i and Q_j are in general position by the technique "push and pull back". Hence the intersection of two polyhedral crosscuts is the finite disjoint union of simple closed curves [B2,pg.14]. Therefore we can untangle Q_i and Q_j by push and pull back technique [See B1]. Consequently we get new crosscuts $Q_i^{'}$ and $Q_j^{'}$ such that $Q_i^{'} \cap Q_j^{'} = \emptyset$, $Bd(Q_i) = Bd(Q_i^{'})$ and $Bd(Q_j) = Bd(Q_j^{'})$. □

LEMMA 3.2. *Let $\epsilon > 0$ and $\{Q_i\}_{i=1}^n$ a finite number of polyhedral cross cuts corresponding to ϵ in the definition of pseudo general polyhedral prime end structure. Let Q_i and Q_j be crosscuts such that $Bd(Q_i) \cap Bd(Q_j) \neq \emptyset$ and even number of points. Then, there exist polyhedral crosscuts $Q_i^{'}$ and $Q_j^{'}$ such that (1) $Q_i^{'} \cap Q_j^{'}$ is disjoint union of polyhedral arcs and (2) $Bd(Q_l) = Bd(Q_l^{'})$ for $l = i, j$.*

Proof. Note that we may consider the intersection of Q_i and Q_j is a finite disjoint union of arcs and simple closed curves by the technique "push and pull back". We now remove simple closed curves also by the technique "push and pull back". If we remove simple closed curves then the intersection of adjusted crosscuts $Q_i^{'}$ and $Q_j^{'}$ is disjoint union of polyhedral arc and $Bd(Q_l) = Bd(Q_l^{'})$ for $l = i, j$. □

M. Brown[Br] defined a local collar and a collar in metric spaces and proved that a locally collared subset in a metric space is collared. This paper has been a critical role in study of 3-manifolds. For example, he shows that a manifold with boundary has collared boundary and two sided $(n-1)$-manifold imbedded in a locally flat fashion in an n-manifold is bi-collared[Br]. Let B be a subset of a topological space X. Then B is *collared* in X if there exists a homeomorphism h carrying $B \times I'$ onto a neighborhood of B such that $h(b, 0) = b$ for all $b \in B$. If B can be covered by a collection of open subsets (relative to B), each of which is collared in X, then B is *locally collared* in X.

LEMMA 3.3. *Let Q_i be a polyhedral crosscut of U and U_i the complementary domain of Q_i. Let $Q_i = D_i \cup_{id.} [Bd(D_i), S_i]$. Then there exists a collar $g_i : D_i \times [0, 1) \to Cl(U_i)$ carrying $D_i \times [0, 1)$ onto a neighborhood of D_i such that $g_i(Bd(D_i) \times [0, 1)) \subset [Bd(D_i), S_i)$.*

Notation: $[Bd(D_i), S_i]$ is a cylinder along Q_i whose ends are $Bd(D_i)$ and S_i and $[Bd(D_i), S_i)$ is half open cylinder.

Proof. We modify crosscut Q_i with the union of D_i and $[Bd(D_i), S_i]$. Then we give a natural collar on $Bd(D_i)$ along $[Bd(D_i), S_i]$. For example, let $h_i : S_i \times I \to Q_i$ be a homotopy such that $h_i(x, 0) = x$ and $h_i(x, 1) = x_0$ for some $x_0 \in D_i$ and $h_i(S_i \times [1/2, 1]) = D_i$.

Recall that the disk D_i is collared in ϵ-neighborhood, i.e., there exists a homeomorphism $g_i : D_i \times [0, 1) \to Cl(U_i)$. We also can take $g_i(x, t) = h_i(x, (1 - t)/2$ on $Bd(D_i) \times [0, 1)$. □

PROPOSITION 3.1. *Let $\epsilon > 0$ and $\{Q_i\}_{i=1}^n$ a finite number of crosscuts corresponding to ϵ in the definition of pseudo general polyhedral prime end structure. Then there exists a 2 sphere S induced by crosscuts $\{Q_i^{''}\}_{i=1}^n$ such that S has a collar in ϵ-neighborhood induced by the local collar as in Lemma 3.3.*

Proof. We will prove this proposition by inductive step.

Let Q_1, Q_2 be crosscuts such that $Bd(Q_1) \cap Bd(Q_2) \neq \emptyset$. We can find such Q_2 by (3) of definition of prime end structure. Then by Lemma 3.2, we can find $Q_2^{'}$ which is pseudo general position with Q_1 such that $(Q_1 \cup Q_2^{'}) - (U_1 \cup U_2^{'})$ forms a 2-manifold with boundary, we denote it with N_1.

Take Q_3 so that $N_1 \cap Q_3 \neq \emptyset$. Then we apply Lemma 3.2 on Q_3 and N_1 and get $Q_3^{'}$ such that N_1 and $Q_3^{'}$ are in pseudo general position. Then $(N_1 \cap Q_3^{'}) - U_3^{'}$ forms a 2-manifold with boundary, denoted by N_2.

Now we suppose that we have N_k which is a 2-manifold with boundary such that $Bd(N_k)$ has diameter so small, i.e., each component of $Bd(N_k)$ is contained in a complementary domain U_i of a crosscut Q_i for some i. Then by the same argument as the above we can find a 2-manifold without boundary without handle, denoted by S'.

Finally we will find a 2-sphere which has a collar induced by the local collar as in Lemma 3.3. In the above we found a sphere S'. But we can not guarantee S' has a collar induced by local collar on D_i's as in Lemma 3.3.

So we take regular neighborhood $\mathcal{R}$ of 1-skeleton $K \subset S'$ in ϵ-neighborhood which are the intersection of Q_i's in S'. We now replace the neighborhood of 1-skeleton in S', $S' \cap \mathcal{R}$, with $Bd(\mathcal{R}) \cap \cup_{i=1}^{i=n} Cl(U_i)$. We denote the modified sphere with S and modified crosscuts with $Q_i^{''}$. Let $D_i'' = S \cap Q_i''$. Then S is covered by $\{D_i''\}_{i=1}^{n}$. Moreover $Bd(D_i'')$ is homeomorphic to S_i and has a collar along Q_i''. This proves the proposition. $\square$

We now state and prove one of our main theorem.

Theorem 3.1. *If U has a pseudo general polyhedral prime end structure, then there exists a C-map $h : U \to IntB^3$.*

Proof. Let $\{\epsilon_n\}$ be a decreasing sequence of positive real number with $\epsilon_i \to 0$. For $\epsilon_1 > 0$, there exists a finite number of cross cuts $\{Q_{1i}\}_{i=1}^{n_1}$ satisfying the definition of pseudo general prime end structure. Therefore we can find a 2-sphere S_1 induced by modified crosscuts $\{Q'_{1i}\}_{i=1}^{n_1}$ as in Proposition 3.1. (Here, Q'_{1i} is Q''_{1i} in Proposition 3.1. Recall that $\cup_{i=1}^{n_i} Bd(D_{1i})$ has a collar p_1 along crosscuts on S_1 such that $p_1(\cup_{i=1}^{n_1} Bd(D'_{1i}) \times 1) = \cup_{i=1}^{n_i} S_{1i}$.

Take ϵ_k and a finite number of crosscuts $\{Q_{ki}\}_{i=1}^{n_k}$ so that ϵ_k-neighborhood induced by $\{Q_{ki}\}_{i=1}^{n_k}$ is contained in complementary domain of S_1 and

$S_1 \cap S_k \neq \emptyset$ where S_k is a 2 sphere induced by $\{Q_{ki}\}_{i=1}^{n_k}$. For convenience we may consider $\epsilon_k = \epsilon_2$.

We now encounter one obstruction to prove theorem. In fact, we can find a collar p_2 on S_2 such that $p_2(\cup_{i=1}^{n_2} Bd(D'_{2i}) \times 1) = \cup_{i=1}^{n_2} S_{2i}$. But we can not guarantee that the collaring structures p_1 and p_2 are compatible. I.e., $p_1(x \times [0,1)) \cap U - E_2$ is in a single fiber of p_2 where E_2 is the open 3 ball whose boundary is S_2.

We now add new crosscuts induced by $Q_{1i} \cap Q_{2j}$ to $\{Q_{2i}\}_{i=1}^{2_n}$ for $i = 1, ..., n_1$ and $j = 1, ..., n_2$. Then new collection of crosscuts $\{Q_{2i}\}_{i=1}^{2_{n'}}$ covers $\cup_{i=1}^{n_1} S_{1i}$. We may consider $\{Q_{2i}\}_{i=1}^{2_{n'}}$ is pseudo general polyhedral since we can modify new crosscuts induced by $Q_{1i} \cap Q_{2j}$ so that new crosscuts is pseudo general position with Q_{2i} by fixing the part of S_{1i}. Consequently, by Proposition 3.1, we can find 2 sphere S_2 such that S_2 has a collar induced by local collar D'_{2i} for $i = 1, ..., 2_{n'}$.

Recall that $\{S_{2i}\}_{i=1}^{2_{n'}}$ covers $\{S_{1i}\}_{i=1}^{1_n}$. Moreover $\cup_{i=1}^{2_{n'}} Bd(D'_{2i})$ has a collaring structure p_2 inherit the collaring structure of $\cup_{i=1}^{1_n} Bd(D'_{1i})$, i.e., a fiber $p_1(x \times [0,1]) \cap U - E_2$ is a fiber of the collar p_2 for $x \in Bd(D'_i)$ where E_2 is the open 3 ball whose boundary is S_2.

We now extend collar $p_1 : S_1 \times [0,1) \to [S_1, S_2)$ to $p_1 : S_1 \times [0,1] \to [S_1, S_2]$ so that $p_1(S_1 \times 1) = p_2(S_2 \times 0)$.

If we apply the above argument to ϵ_2 and ϵ_3 then we get a 2 sphere S_3 corresponding to ϵ_3 and a collar p_3. Moreover $p_2 : S_2 \times [0,1] \to [S_2, S_3]$ is the collar such that $p_2(S_2 \times 1) = p_3(S_3 \times 0)$.

We paste p_1 and p_2 to get a map

$$f_1(x) = \begin{cases} p_1(x) & \text{for } x \in [S_1, S_2] \\ p_2(x) & \text{for } x \in [S_2, S_3]. \end{cases}$$

By continuing this process, we can define $f = \lim_{i\to\infty} f_i$. Then $f : S_1 \times [0,1) \to U - E_1$, where E_1 is the open 3 ball whose boundary is S_1, is the collar blocked by crosscuts $\{Q_{ki}\}_{k=1,i=1}^{k=\infty.i=k_n}$. Let $g : S_1 \times [0,1) \to S_1 \times [0,1/2)$ be a map with $g(x,t) = (x, (1/2)t)$. Consider $f \circ g \circ f^{-1} : U - E_1 \to E^* - E_1$ where E^* is the open 3 ball whose boundary is $f(S_1 \times 1/2)$ which is 2 sphere.

We now extend $h = f \circ g \circ f^{-1}$ to U so that $h = \mathit{identity}$ on E_1 to get a map $h : U \to Int(B^3)$ where $B^3 = f(S_1 \times 1/2) \cup E^*$. Then, by the construction of the map h, it is clear that h is a C-transformation. This proves the theorem. $\square$

Theorem 3.2. *If U has a pseudo polyhedral prime end structure, then U satisfies the induced homeomorphism theorem.* [B-L] *I.e., if $f : Cl(U) \to Cl(U)$ is an onto homeomorphism, then $hfh^{-1} : Int(B^3) \to Int(B^3)$ can be extended to the homeomorphism of B^3 onto itself, where h is the homeomorphism in Theorem 3.1.*

References

[B1] R. H. Bing. *A Surface is Tame if its Complement is 1-ULC.* Trans. Amer. Math. Soc., vol. 101(1961), 294 - 305.

[B2] R. H. Bing, *The Geometric Topology of 3-manifolds* (Amer. Math. Soc. Colloquium Publications, Providence RI, 1983).

[B-L] B. L. Brechner and J. S. Lee, *A Three Dimensional Prime End Theory*, Topology Proceedings, Vol 20 (1995), pp 15-47.

[Br] M. Brown, *Locally flat imbeddings of topological manifolds,* Annals of Math. 75 No. 2, (1962) 331-341.

[C] C. Caratheodory, *Uber die Begrenzung einfach zusammenhangender Gebiete,* Math. Ann. 73 (1913) 323-370.

[E] C. H. Edwards, Jr., *Open 3-manifolds which are simply connected at infinity,* Proc. Amer. Math. Soc. 14 (1963) 391-395.

[Hu] L. Husch, *Finding a boundary for a 3-manifold,* Proc. Amer. Math. Soc. 21 (1969) 64-68.

[Kn] H. Kneser. *Geschlossene Flachen in Dreidemensionalen Mannigfaltigkeiten.* Jber. Deutsch. Math.-Vereinig, vol. 38(1929), 248 - 260.

[U-Y] H. D. Ursell and L.C. Young, *Remarks on the theory of prime ends*, Memoirs Amer. Math. Soc. 3 (1951) 1-29.

[W] C. T. C. Wall, *Open 3-manifolds which are 1-connected at infinity,* Quart. J. Math. 16 (1965) 263-268.

Department of Mathematics
Dongguk University
Seoul 100-715, Korea
E-mail: `jsl@dongguk.edu`