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ON A CERTAIN EXTENDED JIANG SUBGROUP

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Abstract. We introduce a subgroup $HJ(f, x_0, G)$ of the fundamental group of a transformation group as a generalization of the Jiang subgroup $J(f, x_0)$ and show some properties of this subgroup.

1.Introduction

F. Rhodes [4] introduced the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G) as a generalization of the fundamental group of a topological space X and showed that $\sigma(X, x_0, G)$ is isomorphic to $\pi_1(X, x_0) \times G$ if (X, G) admits a family of preferred paths at e. B. J. Jiang [3] introduced the Jiang subgroup $J(f, x_0)$ of the fundamental group of a topological space X.

In the same line with D. H. Gottlieb [1], Jiang Bo-Ju [3] defined the trace group $J(f, x_0)$ of cyclic homotopy from a continuous self-map f to f which is also a subgroup of a fundamental group. The Jiang's subgroup $J(f, x_0)$ is very important and interesting in fixed point theory. Jiang proved the following Lemma in [3].

Lemma. $J(f, x_0) \subset Z(f_\pi(\pi_1(X, x_0)), \pi_1(X, f(x_0)))$.

In this paper, we define a subgroup $HJ(f, x_0, G)$ of $\sigma(X, f(x_0), G)$ and prove that main theorem,

$$HJ(f, x_0, G) \subset Z(f_\pi(\sigma(X, x_0, G)), \sigma(X, f(x_0), G))$$

and investigate some other properties.

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2. Preliminaries and main results

Let (X, G) be a transformation group, where X is a path connected space with x_0 as base point. Given any element g of G, a path f of order g with base point x_0 is a continuous map $f : I \to X$ such that $f(0) = x_0$ and $f(1) = gx_0$. A path f_1 of order g_1 and a path f_2 of order g_2 give rise to a path $f_1 + g_1 f_2$ of order $g_1 g_2$ defined by the equations

$$(f_1 + g_1 f_2)(s) = \begin{cases} f_1(2s), & 0 \le s \le \frac{1}{2} \\ g_1 f_2(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Two paths f and f' of the same order g are said to be homotopic if there is a continuous map $F : I^2 \to X$ such that

$$\begin{aligned} F(s,0) &= f(s),\ 0 \le s \le 1, \\ F(s,1) &= f'(s),\ 0 \le s \le 1, \\ F(0,t) &= x_0,\ 0 \le t \le 1, \\ F(1,t) &= gx_0,\ 0 \le t \le 1. \end{aligned}$$

The homotopy class of a path f of order g was denoted by $[f : g]$. Two homotopy classes of paths of different orders g_1 and g_2 are distinct, even if $g_1 x_0 = g_2 x_0$. F. Rhodes showed that the set of homotopy classes of paths of prescribed order with the rule of composition $*$ is a group, where $*$ is defined by

$$[f_1 : g_1] * [f_2 : g_2] = [f_1 + g_1 f_2 : g_1 g_2].$$

This group was denoted by $\sigma(X, x_0, G)$, and was called the fundamental group of (X, G) with base point x_0.

Let f be a self-map of X. A homotopy $H : X \times I \to X$ is called an f-cyclic homotopy [3] if $H(x,0) = H(x,1) = f(x)$. This concept of a topological space is generalized to that of a transformation group. A continuous map $H : X \times I \to X$ is called an f-homotopy of order g if $H(x,0) = f(x)$, $H(x,1) = gf(x)$, where g is an element of G. If H is an f-homotopy of order g, then the path $\alpha : I \to X$ given by $\alpha(t) = H(x_0, t)$ will be called the trace of H. The trace subgroup of f-homotopies of prescribed order is defined by following definition.

Definition. $HJ(f,x_0,G) = \{[\alpha : g] \in \sigma(X, f(x_0), G) \mid$ there exists homotopy $K : X \times I \to X$ such that $K(x,0) = f(x),\ K(x,1) = gf(x)$ and $K(g'x_0,t) = g'\alpha(t)$ for some $g' \in G\}$.

Then K is called by Hf-homotopy with trace α. In particular, if there exists Hf-homotopy H of order g such that $H(x,t)$ is a homomorphism of (X,G), then $[H(x_0,t) : g]$ belongs to $HJ(f,x_0,G)$. Of course $J(f,x_0) \subset HJ(f,x_0,G)$. $HJ(f,x_0,\{e\})$ is also defined by $J(f,x_0)$ in [3]. From this fact, we say that $HJ(f,x_0,G)$ is an extended Jiang subgroup. It is easy to show that this subgroup $HJ(f,x_0,G)$ is a subgroup of $\sigma(X,f(x_0),G)$.

Theorem 1. *If $f : (X,G) \to (X,G)$ is a homomorphism and G is abelian, then $HJ(f,x_0,G) \subset Z(f_\pi(\sigma(X,x_0,G)), \sigma(X,f(x_0),G))$.*

Proof. Let $[\alpha : g]$ be an element $HJ(f,x_0,G)$. Then there exists a f-homotopy $K : X \times I \to X$ of order g such that $K(x,0) = f(x)$, $K(x,1) = gf(x)$ and $K(hx_0,t) = h\alpha(t)$ for some $h \in G$. For some $[\beta, g'] \in \sigma(X,x_0,G)$, we must show that $[\alpha : g] * f_\pi[\beta : g'] = f_\pi[\beta : g'] * [\alpha : g]$. That is, since G is abelian, $\alpha + gf\beta$ is homotopic to $f\beta + g'\alpha$. Let $J : I \times I \to X$ be a homotopy such that $J = K(\beta \times I)$. Define a homotopy $F : I \times I \to X$ by

$$F(s,t) = \begin{cases} J(2s(1-t), 2st), & 0 \le s \le \frac{1}{2} \\ J(1-(2-2s)t, (2-2s)t + 2s - 1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Therefore

$$\begin{aligned} F(s,0) &= \begin{cases} J(2s,0), & 0 \le s \le \frac{1}{2} \\ J(1,2s-1), & \frac{1}{2} \le s \le 1 \end{cases} \\ &= \begin{cases} K(\beta(2s),0), & 0 \le s \le \frac{1}{2} \\ K(g'x_0, 2s-1), & \frac{1}{2} \le s \le 1 \end{cases} \\ &= (f\beta + g'\alpha)(s). \end{aligned}$$

$$\begin{aligned} F(s,1) &= \begin{cases} J(0,2s), & 0 \le s \le \frac{1}{2} \\ J(2s-1,1), & \frac{1}{2} \le s \le 1 \end{cases} \\ &= \begin{cases} K(x_0,2s), & 0 \le s \le \frac{1}{2} \\ K(\beta(2s-1),1), & \frac{1}{2} \le s \le 1 \end{cases} \\ &= (\alpha + gf\beta)(s). \end{aligned}$$

$F(0,t) = J(0,0) = K(x_0,0) = f(x_0)$ and $F(1,t) = J(1,1) = K(g'x_0,1) = gf(g'x_0) = g'gf(x_0) = gg'f(x_0)$. So, $[\alpha + gf\beta : gg'] = [f\beta + g'\alpha : g'g]$. □

THEOREM 2. *If $f, f' : (X,G) \to (X,G)$ are homotopic homomorphisms, then $HJ(f, x_0, G)$ and $HJ(f', x_0, G)$ are isomorphic groups.*

Proof. Let $H : X \times I \to X$ be a homomorphic homotopy from f to f'. Let $P(t) = H(x_0, t)$ for every element t of I . Then P is a path from $f(x_0)$ to $f'(x_0)$. It is sufficient to show that $P_\pi(HJ(f, x_0, G)) \subset HJ(f', x_0, G)$. Let $[\alpha : g]$ be any element of $HJ(f, x_0, G)$. Then there exists a Hf-homomorphic homotopy $G : X \times I \to X$ of order g such that $G(x,0) = f(x)$, $G(x,1) = gf(x)$ and $G(g'x_0, t) = g'\alpha(t)$ for some $g' \in G$. If we define a homotopy $K = H^{-1} \circ G \circ (gH) : X \times I \to X$ given by

$$K(x,t) = \begin{cases} H(x, 1-3t), & 0 \le t \le \frac{1}{3} \\ G(x, 3t-1), & \frac{1}{3} \le t \le \frac{2}{3} \\ gH(x, 3t-2), & \frac{2}{3} \le t \le 1, \end{cases}$$

then $K(x,0) = H(x,1) = f'(x)$, $K(x,1) = gH(x,1) = gf'(x)$ and

$$\begin{aligned} K(g'x_0,t) &= \begin{cases} H(g'x_0, 1-3t), & 0 \le t \le \frac{1}{3} \\ G(g'x_0, 3t-1), & \frac{1}{3} \le t \le \frac{2}{3} \\ gH(g'x_0, 3t-2), & \frac{2}{3} \le t \le 1. \end{cases} \\ &= \begin{cases} g'H(x_0, 1-3t), & 0 \le t \le \frac{1}{3} \\ g'G(x_0, 3t-1), & \frac{1}{3} \le t \le \frac{2}{3} \\ g'gH(x_0, 3t-2), & \frac{2}{3} \le t \le 1. \end{cases} \\ &= g'K(x_0,t) = g'(P^{-1} \circ \alpha \circ (gP))(t) = g'P_\pi(\alpha(t)). \end{aligned}$$

So, $P_\pi(HJ(f, x_0, G))$ is contained in $HJ(f', x_0, G)$. □

COROLLARY 3. *If $f, k : X \to X$ are homotopic, then $J(f, x_0)$ and $J(k, x_0)$ are isomorphic.*

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