Kangweon-Kyungki Math. Jour. **9** (2001), No. 1, pp. 53–60

ON THE POSITIVITY OF MATRICES RELATED TO THE LINEAR FUNCTIONAL

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Abstract. We study the properties of positivity of matrices and construct useful positive matrices. As an application, we consider a directed graph with matrices such that all the associated matrices related to the positive linear functional are positive.

1. Introduction

In this paper, we deal with various positive matrices which come from positive matrices and construct associated positive matrices related to the linear functional on a C^*- algebra. Although the proof of the positivity of matrices related the linear functional may be a folklore for specialists, the authors give direct proofs of the positivity of matrices. In detail, the purpose of this paper is to introduce many positive matrices and to give a construction of positive matrices related to the linear functional on the C^*-algebras by using a graph.

Here we briefly review some definitions and notations which are necessary for our discussions that follow.

As is known, graph theory is useful to the study of C^*-algebras (see [4]) and we use here a directed graph.

A directed graph $\mathcal{G}$ consists of a nonempty set V of vertices, E of edges, and the range, source maps $r, s : E \to V$. So we denote a graph $\mathcal{G}$ by $\mathcal{G} = (V, E, r, s)$. For convenience, we denote an edge $e \in E$ with $s(e) = u$ and $r(e) = v$ by uv. Recall that for any $u \in V$, $s^{-1}(u)$ is the set $\{e \in E | s(e) = u\}$ and a vertex u with $s^{-1}(u) = \emptyset$ is called a sink.

Throughout this paper, $\mathcal{M}_{k,l}$ (resp. $\mathcal{M}_k$) is the set of all $k \times l$ (resp. $k \times k$) matrices over $\mathbb{C}$ and O_k denotes zero matrix in $\mathcal{M}_k$. For a matrix

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Received February 1, 2001.

1991 Mathematics Subject Classification: Primary 46L30, Secondary 46L05.

Key words and phrases: positive matrices, linear functional, Cuntz algebra, directed graph.

$A \in \mathcal{M}_k$, sometimes we use the expression of $A = (A^1, A^2, \cdots, A^k)$, where A^i is the i-th column of A. For any $A \in \mathcal{M}_k$, A is said to be positive if $A = C^*C$ for some $C \in \mathcal{M}_k$, where the adjoint matrix $C^* = (d_{ij})$ of $C = (c_{ij})$ is given by $d_{ij} = \overline{c_{ji}}$. We denote a positive matrix A by $A \geq 0$. Furthermore, for a C^*-algebra $\mathcal{B}$, $\mathcal{M}_k(\mathcal{B})$ denotes the set of all $k \times k$ matrices (x_{ij}) whose elements (x_{ij})'s are in $\mathcal{B}$.

In addition, we recall that the inner product $< \xi, \eta >$ of two vectors $\xi = (\xi^1, \xi^2, \cdots, \xi^n)$ and $\eta = (\eta^1, \eta^2, \cdots, \eta^n)$ in $\mathbb{C}^n$ is given by $< \xi, \eta >= \sum \xi^i \overline{\eta^i}$ and we get $|\xi|^2 =< \xi, \xi >$.

2. Constructions of Positive matrices from Positive matrices

In this section we construct various matrices from positive matrices and give direct proofs that the constructed matrices are positive.

At first, we introduce a positive matrix from a positive matrix and a complex number. For a matrix $A = (a_{ij}) \in \mathcal{M}_k$ and a scalar $\xi \in \mathbb{C}$, we define the matrix $A^\xi \in \mathcal{M}_{k+1}$ as follow:

$$A^\xi = \begin{pmatrix} |\xi|^2 & \xi a_{11} & \xi a_{12} & \cdots & \xi a_{1k} \\ \bar{\xi} a_{11} & a_{11} & a_{12} & \cdots & a_{1k} \\ \bar{\xi} a_{21} & a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{\xi} a_{k1} & a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

LEMMA 2.1. *Let A be any matrix and ξ be any complex number. If A is a positive matrix, then A^ξ is a positive matrix.*

Proof. Let A be a positive matrix in $\mathcal{M}_k$. Then there exists a matrix $B = (B^1, B^2, \cdots, B^k) \in \mathcal{M}_k$ such that $A = B^*B$. For any $\xi \in \mathbb{C}$, if we let C be $(\bar{\xi} B^1, B^1, B^2, \cdots, B^k) \in \mathcal{M}_{k,k+1}$, then by simple calculations we get $C^*C = A^\xi$ which implies that A^ξ is positive. □

The *Hadamad product* $A \circ B$ of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathcal{M}_k$ is defined to be just their elementwise product $A \circ B = (a_{ij} b_{ij}) \in \mathcal{M}_k$.

For $A = (a_{ij}) \in \mathcal{M}_k$, $B = (b_{ij}) \in \mathcal{M}_l$, we define the direct sum $A \oplus B \in \mathcal{M}_{k+l}$ as follow:

$$A \oplus B = \begin{pmatrix} A & O \\ O & B \end{pmatrix}$$

Furthermore, for $A = (a_{ij}) \in \mathcal{M}_k$, $B = (b_{ij}) \in \mathcal{M}_l$, we define a matrix $A \diamond B \in \mathcal{M}_{k+l-1}$ as follow:

$$A \diamond B = \begin{pmatrix} a_{11} + b_{11} & a_{12} & \cdots & a_{1k} & b_{12} & \cdots & b_{1l} \\ a_{21} & a_{22} & \cdots & a_{2k} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} & 0 & \cdots & 0 \\ b_{21} & 0 & \cdots & 0 & b_{22} & \cdots & b_{2l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{l1} & 0 & \cdots & 0 & b_{l2} & \cdots & b_{ll} \end{pmatrix}$$

Simple calculations give us some properties of the above operation $\diamond$. In the following, we present these properties without proofs.

Lemma 2.2. *For any $A \in \mathcal{M}_k$, $B \in \mathcal{M}_l$, and $C \in \mathcal{M}_s$, we have the followings:*

(1) $O_k \diamond O_l = O_{k+l-1}$
(2) $A \diamond B = A \diamond O_l + O_k \diamond B$
(3) $(A \diamond B) \diamond C = A \diamond (B \diamond C)$.

With the notations as above, we obtain the following lemma.

Lemma 2.3. *Let A and B be positive matrices. Then $A \circ B$, $A \oplus B$, and $A \diamond B$ are positive matrices.*

Proof. By elementary calculations, the proofs are immediate and we omit them. □

Corollary 2.4. *For any positive matrix $A \in M_s$, the matrix $O_k \diamond A \diamond O_l$ is positive.*

Proof. From Lemma 2.3, for any positive matrix $A \in M_s$, $O_k \diamond A \geq 0$ holds. The fact of $O_l \geq 0$ and Lemma 2.3 give $O_k \diamond A \diamond O_l \geq 0$. □

For $A = (a_{ij}) \in \mathcal{M}_k$, we now define the matrix $A^{11} \in \mathcal{M}_k$ as follow:

$$A^{11} = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

THEOREM 2.5. *Let $A_i,\ i = 1, 2, \cdots, k$ be positive matrices. For any $\xi_i \in \mathbb{C},\ i = 1, 2, \cdots, k$ with $\sum_{i=1}^{k} |\xi_i|^2 \leq 1$, we have*

$$(A_1^{\xi_1} \diamond A_2^{\xi_2} \diamond \cdots \diamond A_k^{\xi_k})^{11} \geq 0.$$

Proof. By definitions, we get that $A_1^{\xi_1} \diamond A_2^{\xi_2} \diamond \cdots \diamond A_k^{\xi_k}$ is a matrix whose $(1, 1)$-component is $\sum_{i=1}^{k} |\xi_i|^2$.

For any ξ_i and positive matrix A_i, $i = 1, 2, \cdots, k$, Lemma 2.1 gives that $A_i^{\xi_i}$ is positive. On the other hand, by Lemma 2.3,

$$A_1^{\xi_1} \diamond A_2^{\xi_2} \diamond \cdots \diamond A_k^{\xi_k}$$

is also a positive matrix.

Here we note that for a matrix $A = (a_{ij}) \in \mathcal{M}_k$, we have

$$A^{11} = A + (1 - a_{11})E_{11},$$

where E_{11} is a matrix whose $(1, 1)$-component is 1 and the others are 0. Thus, if A is a positive matrix with $a_{11} \leq 1$, then we get $A^{11} \geq 0$.

Therefore, the fact $\sum_{i=1}^{k} |\xi_i|^2 \leq 1$ gives that $(A_1^{\xi_1} \diamond A_2^{\xi_2} \diamond \cdots \diamond A_k^{\xi_k})^{11}$ is positive. □

3. Positivity of the associated matrices related to the linear functional

In this section, we determine a graph with positive matrices related to positive linear functional on a C^*-algebra. In detail, we consider the Cuntz algebra together with the Cuntz state. We construct a graph with positive matrices attached to each vertices and each edges which is related to the positive linear functional on the Cuntz algebra. In other words, for a given set of monomials in the Cuntz algebra, we define matrices related to the Cuntz state and we use graph theory to show that these matrices are positive.

For $n = 2, 3, \cdots$, let $\mathcal{B}$ be a simple infinite C^*-algebra generated by n isometries (see [2]). We note that an element X in $\mathcal{B}$ which consists of k isometries is called a *monomial* with length k. For two monomials X and Y in $\mathcal{B}$, we denote $X < Y$ if $Y = XZ$ for some non-identity monomial Z in $\mathcal{B}$.

As is known, a positive linear functional on a C^*-algebra is completely positive. So when ρ is a positive linear functional on the Cuntz algebra $\mathcal{B}$, for any $k \in \mathbb{N}$ and a positive matrix $(x_{ij}) \in \mathcal{M}_k(\mathcal{B})$, the linear functional ρ_k on $\mathcal{M}_k(\mathcal{B})$ which is defined by $\rho_k((x_{ij})) = (\rho(x_{ij})) \in \mathcal{M}_k$ is also positive.

At first, for a given set of monomials in $\mathcal{B}$, we define a matrix over $\mathbb{C}$ related to such a linear functional ρ.

Let ρ be a linear functional on the Cuntz algebra $\mathcal{B}$. For any $k \in \mathbb{N}$ and any monomials $X_1, X_2, \cdots, X_k \in \mathcal{B}$, consider the matrix $(\rho(X_i^* X_j)) \in \mathcal{M}_k$.

In the following proposition, we show that the positive linear functional ρ on $\mathcal{B}$ gives a matrix which is positive.

PROPOSITION 3.1. *If ρ is the positive linear functional on the Cuntz algebra $\mathcal{B}$. Then for any $k \in \mathbb{N}$ and monomials $X_1, X_2, \cdots, X_k$ in $\mathcal{B}$, the matrix $(\rho(X_i^* X_j)) \in \mathcal{M}_k$ is positive.*

Proof. Let ρ be a linear functional on the Cuntz algebra $\mathcal{B}$.

For any $k \in \mathbb{N}$ and monomials $X_1, X_2, \cdots, X_k$ in $\mathcal{B}$, the fact of

$$(X_i^* X_j) = (X_1, \cdots, X_k)^* (X_1, \cdots, X_k) \in \mathcal{M}_k(\mathcal{B}),$$

gives that the matrix $(X_i^* X_j)$ is positive. If ρ is a positive linear functional and so it is completely positive, then we have that $\rho_k((X_i^* X_j))$ is a positive matrix. Thus the matrix $(\rho(X_i^* X_j)) = \rho_k((X_i^* X_j))$ is also positive. □

From now on, we consider the positive linear functional ρ on the C^*-algebra $\mathcal{B}$ (see [1], [3]).

Now, we construct a graph whose vertex set is a set of monomials in $\mathcal{B}$. Furthermore, we assign matrices to each vertices and to each edges such that all matrices are positive.

For any set $\{X_1, X_2, \cdots, X_k\}$ of monomials in $\mathcal{B}$, We determine a directed graph $\mathcal{G} = (V, E, r, s)$ with $V = \{X_1, X_2, \cdots, X_k\}$ and $E = \{X_iX_j | X_i < X_j, i, j = 1, \cdots, k\}$.

Now for this directed graph $\mathcal{G} = (V, E, r, s)$, we construct matrices associated to each vertex in V and each edge in E.

We define matrices $M(X_i)$ and $N(X_iX_j)$ in $\cup_{k\in\mathbb{N}}\mathcal{M}_k$ associated to each vertex $X_i \in V$ and each edge $X_iX_j \in E$, respectively, as follows:

(1) When X_i is a sink, the matrix $M(X_i)$ associated to X_i is (1).

(2) When X_j is a sink and $\xi = \rho(X_i^*X_j)$, the matrix $N(X_iX_j)$ associated to X_iX_j is $\begin{pmatrix} |\xi|^2 & \xi \\ \bar{\xi} & 1 \end{pmatrix}$. Generally,

$$N(X_iX_j) = M(X_j)^{\xi},$$

where $M(X_j)$ is the matrix associated to $X_j \in V$ and $\xi = \rho(X_i^*X_j)$.

(3) When $s^{-1}(X_i) = \{X_iX_{j_1}, X_iX_{j_2}, \cdots, X_iX_{j_k}\}$,

$$M(X_i) = (N(X_iX_{j_1}) \diamond N(X_iX_{j_2}) \diamond \cdots \diamond N(X_iX_{j_k}))^{11}.$$

Since $\mathcal{G}$ is a directed graph, by repeating above three steps, we can associate a matrix to each vertex and each edges.

Theorem 3.2. *With the notations as above, for any vertex $X_i \in V$ and edge $X_iX_j \in E$, the matrices $M(X_i)$ and $N(X_iX_j)$ are positive.*

Proof. Trivially, $(1) \geq 0$ and $(1)^{\xi} \geq 0$ hold.

For any $X_i \in V$ with $s^{-1}(X_i) \neq \emptyset$, we have

$$\sum_{X_iX_j \in s^{-1}(X_i)} |\rho(X_i^*X_j)|^2 \leq 1.$$

Thus, by Lemma 2.1, Lemma 2.3, and Lemma 2.5, we conclude that the matrices $M(X_i)$ and $N(X_iX_j)$ associated to each vertex $X_i \in V$ and edge $X_iX_j \in E$ are positive. □

As an example, now we construct a graph which comes from a set of monomials of a C^*- algebra with a positive linear functional.

Example 3.3. Let $\{X_1, X_2, X_3, X_4\}$ be the set of monomials in the Cuntz algebra $\mathcal{B}$ with Cuntz state ρ satisfying $X_1 < X_2 < X_3$, $X_1 < X_2 < X_4$, and $X_3^* X_4 = 0$. First we construct a directed graph $\mathcal{G} = (V, E, r, s)$ which is defined by the set $\{X_1, X_2, X_3, X_4\}$. Let V be the set $\{X_1, X_2, X_3, X_4\}$ and for two monomials X_i and X_j with $X_i < X_j$, there exists an edge $X_i X_j \in E$. Thus we have $E = \{X_1X_2, X_2X_3, X_2X_4\}$ and complex numbers $\rho(X_1^* X_2)$, $\rho(X_2^* X_3)$, and $\rho(X_2^* X_4)$. Then the matrices $M(X_3)$ and $M(X_4)$ associated to X_3 and X_4, respectively, are (1). The matrices $N(X_2X_3)$ and $N(X_2X_4)$ associated to X_2X_3 and X_2X_4 are $\begin{pmatrix} |b|^2 & b \\ \bar{b} & 1 \end{pmatrix}$ and $\begin{pmatrix} |c|^2 & c \\ \bar{c} & 1 \end{pmatrix}$, respectively, where $b = \rho(X_2^* X_3)$ and $c = \rho(X_2^* X_4)$. The matrix $M(X_2)$ associated to X_2 and the final matrix $M(X_1)$ are

$$M(X_2) = \begin{pmatrix} 1 & b & c \\ \bar{b} & 1 & 0 \\ \bar{c} & 0 & 1 \end{pmatrix} \quad \text{and} \quad M(X_1) = \begin{pmatrix} 1 & a & ab & ac \\ \bar{a} & 1 & b & c \\ \bar{c} & \bar{b} & 1 & 0 \\ \bar{a}\bar{c} & \bar{c} & 0 & 1 \end{pmatrix},$$

where $a = \rho(X_1^* X_2)$.

It is straightforward to show directly that all matrices above are positive. On the other hand, Theorem 3.2 allows us that all matrices above are positive.

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