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THE DOMINATION NUMBER OF A TOURNAMENT

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Abstract. We find bounds for the domination number of a tournament and investigate the sharpness of these bounds. We also find the domination number of a random tournament.

1. Introduction

Let D be a digraph. A subset S of the vertex set $V(D)$ is a *dominating set* of D if for each vertex v not in S there exists a vertex u in S such that uv is an arc of D. Note that $V(D)$ itself is a dominating set of D. A dominating set of D with the smallest cardinality is called a *minimum dominating set* of D and its cardinality is the *domination number* of D. We will reserve $\alpha(D)$ for the domination number of D. For subsets S and T of $V(D)$, we say that S *dominates* T if for every $v \in T$ there exists $u \in S$ such that uv is an arc of D. For definitions and notation not given here see [1] and [6].

A *tournament* is a digraph in which every pair of distinct vertices has exactly one arc. A *transitive* tournament is a tournament such that if uv and vw are arcs then uw is also an arc.

Let us consider the probability space $\mathcal{T}_n$ consisting of random tournaments on the vertex set $V = \{1, 2, \ldots, n\}$. By a *random tournament* we mean here a tournament on V obtained by choosing, for each $1 \le i < j \le n$, independently, either the arc ij or the arc ji, where each of these two choices is equally likely. Observe that all the $2^{\binom{n}{2}}$ possible tournaments on V are equally likely.

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In section 2 we show that

$$1 \le \alpha(T) \le \lfloor \lg(n+1) \rfloor$$

for any tournament T of order n. Here lg denotes the logarithm with base 2. In section 3 we show that a random tournament $T \in \mathcal{T}_n$ has domination number either

$$\lfloor k_* \rfloor + 1 \quad \text{or} \quad \lfloor k_* \rfloor + 2,$$

where $k_* = \lg n - 2 \lg\lg n + \lg\lg e$ and investigate the sharpness of the upper bound for the domination number of a tournament.

2. Tournaments

In this section we find bounds for the domination number of a tournament.

First we introduce an algorithm which finds a dominating set of a given tournament. This algorithm is greedy in the sense that it selects a vertex that covers a maximum number of yet uncovered vertices in each step.

ALGORITHM. *Let $T_1 = T$ be a given tournament of order n and let $S_0 = \emptyset$. Put $i = 1$ and go to* (1).

(1) *Choose a vertex v_i with largest outdegree in T_i and let $S_i = S_{i-1} \cup \{v_i\}$.*

(2) *Let T_{i+1} be the subtournament of T_i induced by $V(T_i) - N^+_{T_i}[v_i]$.*

(3) *If T_{i+1} is an empty tournament, then let $S = S_i$ and stop. Otherwise, put $i = i + 1$ and return to* (1). □

We note that the complexity of this algorithm is $\mathcal{O}(n^2)$. But we will see shortly that this estimate can be improved.

Let T be a tournament of order n. Then we know that there exists a vertex v in T with $od(v) \ge (n-1)/2$ since $\sum_{v \in V} od(v) = n(n-1)/2$ and hence the average outdegree over all vertices is $(n-1)/2$. In addition, every subdigraph of a tournament induced by a subset of $V(T)$ is also a tournament.

Using these simple observations, we prove the following theorem.

THEOREM 1. *Let T be a tournament of order n. Then algorithm above terminates in at most $\lfloor \lg(n+1) \rfloor$ steps and S is a dominating set for T. Therefore we have*

$$1 \leq \alpha(T) \leq \lfloor \lg(n+1) \rfloor,$$

where lg *denotes the logarithm with base* 2.

Proof. Step 1: Let $T_1 = T$ and choose a vertex v_1 of T_1 having maximum outdegree.

Step 2: Let T_2 be the subtournament of T_1 induced by $V(T_1) - N^+_{T_1}[v_1]$. Since

$$|N^+_{T_1}[v_1]| \geq \frac{n-1}{2} + 1 = \frac{n+1}{2},$$

we have

$$n_2 := |V(T_2)| = n - |N^+_{T_1}[v_1]| \leq \frac{n-1}{2}.$$

Choose a vertex v_2 of T_2 having maximum outdegree.

Step 3: Let T_3 be the subtournament of T_2 induced by $V(T_2) - N^+_{T_2}[v_2]$. Since

$$|N^+_{T_2}[v_2]| \geq \frac{n_2+1}{2},$$

we have

$$n_3 := |V(T_3)| = n_2 - |N^+_{T_2}[v_2]| \leq \frac{n_2-1}{2} \leq \frac{n-(1+2)}{2^2}.$$

Choose a vertex v_3 of T_3 having maximum outdegree. We continue this process up to step k.

Step k: Let T_k be the subtournament of T_{k-1} induced by $V(T_{k-1}) - N^+_{T_{k-1}}[v_{k-1}]$. Then

$$\begin{aligned} n_k := |V(T_k)| &= n_{k-1} - |N^+_{T_{k-1}}[v_{k-1}]| \\ &\leq \frac{n_{k-1}-1}{2} \\ &\leq \frac{n-(2^0+2^1+\cdots+2^{k-2})}{2^{k-1}}. \end{aligned}$$

Choose a vertex v_k of T_k having maximum outdegree.

After step k, the number of vertices in T that are not yet covered by $\{v_1, v_2, \dots, v_k\}$ is

$$\begin{aligned} n_k - |N^+_{T_k}[v_k]| &\le \frac{n_k - 1}{2} \\ &\le \frac{n - (2^0 + 2^1 + \cdots + 2^{k-1})}{2^k}. \end{aligned} \tag{1}$$

We want to find the minimum value k' of k that makes (1) zero. It is easy to see that $k' \le \lg(n+1)$. Clearly, $S = \{v_1, v_2, \dots, v_{k'}\}$ is a dominating set of T. □

Now we can see from Theorem 1 that the complexity of the algorithm is $\mathcal{O}(n \log n)$.

We will discuss the sharpness of the upper bound in the above theorem later. The lower bound is sharp. Any transitive tournament will do.

It is easily seen that every tournament is unilateral and that every strong tournament has at least three vertices.

Corollary. *Let T be a strong tournament of order n. Then we have*

$$2 \le \alpha(T) \le \lfloor \lg(n+1) \rfloor.$$

Moreover, the lower bound is sharp.

Proof. We know that a tournament is strong if and only if there exists a spanning cycle of the tournament [4, p. 306]. Therefore any strong tournament T of order n has no vertices of outdegree $n-1$ and so $\alpha(T) \ge 2$. For the sharpness of the lower bound, we construct a tournament T as follows. Take an n-cycle C_n and let v be a fixed vertex of C_n. Join v to all possible vertices of C_n and choose the other arcs arbitrarily. Then the resulting tournament T is strong since it has a spanning cycle, and $\alpha(T) = 2$. □

3. Random Tournaments and Paley Tournament

In this section we find the domination number of a random tournament and investigate the sharpness of the upper bound for the domination number of a tournament found in section 1.

THEOREM 2. *A random tournament $T \in \mathcal{T}_n$ has domination number either*

$$\lfloor k_* \rfloor + 1 \quad \text{or} \quad \lfloor k_* \rfloor + 2,$$

where $k_* = \lg n - 2 \lg \lg n + \lg \lg e$ and $\lg$ *denotes the logarithm with base* 2.

Proof. Let X be a nonnegative random variable such that $X(T)$ is the number of dominating k-sets of T for each $T \in \mathcal{T}_n$. If K is a fixed k-set of vertices, then the probability that K does not dominate a fixed vertex in $V - K$ is

$$Prob(K \text{ does not dominate a fixed vertex in } V - K) = 2^{-k}.$$

Hence, the probability that K dominates a fixed vertex in $V - K$ is

$$Prob(K \text{ dominates a fixed vertex in } V - K) = 1 - 2^{-k}$$

and the probability that K dominates all vertices in $V - K$ is

$$Prob(K \text{ dominates all vertices in } V - K) = (1 - 2^{-k})^{n-k}.$$

Therefore, the expectation $E[X]$ of the random variable X is

$$E[X] = \binom{n}{k} (1 - 2^{-k})^{n-k}.$$

The rest of this proof is exactly the same as the proof in [5] or [7] once we take $r = 2$. □

REMARK. Now let us consider the sharpness of the upper bound in Theorem 1. Theorem 2 says that not only do tournaments of order n having domination number $(1+o(1)) \lg n$ exist, but when n is large, the overwhelming majority of tournaments will have a domination number near $\lg n$. Can we construct such a tournament?

The proof of Theorem 1 strongly suggests that a quasi-random tournament has a large domination number (see [2]). Then do quasi-random tournaments really have domination number very close to the upper bound $\lfloor \lg(n + 1) \rfloor$ for n sufficiently large? A well-known example of a quasi-random tournament is so-called *Paley tournament*

$Q_p(\mathbb{Z}_p, E)$. For a prime $p \equiv 3 \pmod 4$, the vertices of Q_p consist of integers modulo p. An ordered pair $(i,j) \in E$ if and only if $i-j$ is a non-zero quadratic residue modulo p, i.e., if and only if $(\frac{i-j}{p}) = 1$, where we use the familiar Legendre symbol. Then Q_p is a well-defined $(p-1)/2$-regular quasi-random tournament (see [2]). It is easily checked that $\alpha(Q_p) = \lfloor \lg(p+1) \rfloor$ for $p = 3, 7, 11, 19$. But $\alpha(Q_{31}) \le 4 < \lfloor \lg(31+1) \rfloor$ since $\{1,2,4,5,7,8,9,10,14,16,18,19,20,25,28\}$ is the set of all non-zero quadratic residues modulo 31 and hence $\{0,27,29,30\}$ is a dominating set for Q_{31}. This shows that $\alpha(Q_p) = \lfloor \lg(p+1) \rfloor$ does not hold for some p. What if p is large enough? Now we consider Schütte property. We say that a tournament has (*Schütte*) *property* S_k if for every set of k vertices there is one vertex that dominates them all. For example, a directed 3-cycle has property S_1.

The following lemma appears in [3], but it was used to find a lower bound of p for Q_p to have property S_k.

Lemma. *If k satisfies the inequality*

$$p - \{(k-2)2^{k-1} + 1\}\sqrt{p} - 2^{k-1} > 0,$$

then Paley tournament Q_p has property S_k. □

Now we are ready to state the following theorem.

Theorem 3. *The domination number of Paley tournament Q_p satisfies*

$$\alpha(Q_p) > (1 + o(1))\frac{1}{2}\lg p.$$

Proof. Suppose that Paley tournament Q_p satisfies property S_k. Then for every set S of k vertices, there exists a vertex not in S that is dominated by S and hence every dominating set must have more than k vertices. Consequently, $\alpha(Q_p) > k$ if Q_p satisfies property S_k.

Now we know that Q_p satisfies S_k if

$$\{(k-2)2^{k-1} + 1\}\sqrt{p} + 2^{k-1} < p. \tag{2}$$

Hence we want to find the maximum value k' of k satisfying (2) when p is large. But it is easy to check $k' < \lg(p+1)$ and so we let

$$k = c\lg p - d \lg\lg p + 1, \qquad c > 0 \quad \text{and} \quad d \ge 0.$$

Then the left side of (2) becomes

$$p\left\{\frac{p^{c-1/2}}{(\lg p)^d}\lg\left(\frac{p^c}{2(\lg p)^d}\right)+\frac{1}{\sqrt{p}}+\frac{p^{c-1}}{(\lg p)^d}\right\}. \tag{3}$$

To make the second factor of (3) smaller than 1 when $p\to\infty$, we must have $c\le 1/2$. But the maximum value k' of k can be obtained when $c=1/2$ and $d>0$. Therefore

$$k'=\frac{1}{2}\lg p-d\lg\lg p+1,\quad d>0$$

and so

$$\alpha(Q_p)>k'=(1+o(1))\frac{1}{2}\lg p.$$

□

We do not know yet whether our upper bound in Theorem 1 is sharp and hence two natural questions now arise.

Open Questions. (1) We have shown that a tournament T of order n has domination number

$$\alpha(T)\le\lfloor\lg(n+1)\rfloor$$

in Theorem 1. Can we either sharpen this upper bound or construct a tournament of order n whose domination number is this upper bound?

(2) Can we find the domination number of Paley tournament Q_p as a function of p? What about the asymptotics for the domination number of Q_p?

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