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최대 부호화 이득을 내는 웨이블릿 기저를 구축하기 위한 고속 알고리즘

Fast Algorithm for Constructing Wavelet Packet Bases Yielding the Maximum Coding Gain

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요 약

본 논문에서는 주어진 필터와 구현 복잡도에 대하여 최대 코딩이득을 내는 부 주파수 분할을 가진 서브밴드 부호화기를 구축하는 고속 알고리듬을 제안한다. 이를 위하여 본 논문에서는 직교 기저 및 비 직교 기저와 임의의 부 주파수 분할에 대하여 적용할 수 있는 통합적인 코딩이득의 식을 유도한 다음, 부 주파수 대역 수에 대하여 코딩이득이 단순 증가 함수임을 증명한다. 이를 바탕으로 복잡도에 대하여 최대 코딩이득을 내는 최적화 된 부 주파수 분할을 찾아내기 위하여 그 단순 증가 함수를 부 주파수 대역 수에 따른 왜곡 함수로 다룬다. 이 왜곡 함수을 목적함수로 두고 Lagrange 방법에 근거하여 최적화 된 해를 고속으로 제공하는 알고리듬을 개발한다.

Abstract

This paper develops the fast dynamic programming technique to construct the subband structure yielding the maximum coding gain for given filter bases and a given limit of implementation complexity. We first derive the unified coding gain which can be applied to non-orthogonal filter basis as well as orthogonal filter basis and to arbitrary subband decompositions. Then, we verify that the unified coding gains in real systems are monotonically increasing function for the implementation complexities which are proportional to the number of subbands. By using this phenomenon, the implementation complexity and the coding gain are treated in the same way as the rate and distortion function. This makes it possible to use the Lagrangian multiplier method for finding the optimal subband decomposition producing the maximum coding gain for a given limit of implementation complexity.

I. Introduction

Subband coding (SBC) has often been used for signal compression as a valuable tool. SBC decomposesa signal frequency space into groups of subbands in order to increase the signal energy compaction. SBC distributes the given coding bits economically among subbands, because subband signals with low energy are encoded by lower quantization levels or discarded altogether.

The coding gain evaluates the performance of an SBC. The coding gain is defined as the ratio of the input signal variance to the geometric mean of the subband signal variances. The coding gain

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actually compares an SBC with the direct quantization scheme called Pulse Code Modulation (PCM). Since the gain is obtained under the assumption that the quantizer bits are the most efficiently allocated, it directly reflects the signal energy compaction property of filter banks used to implement the SBC and the subband decomposition of the SBC. Therefore, an SBC should be designed to have optimal filter banks and optimal subband decompositions, yielding the maximum coding gain.

This paper first derives the unified coding gain which works for any structure of subband decompositions and any kind FIR filters including linearly phased filter as well as orthogonal filter, and then the paper develops the fast dynamic programming technique to search for the optimal subband decomposition within the allowed limit of implementation complexity.

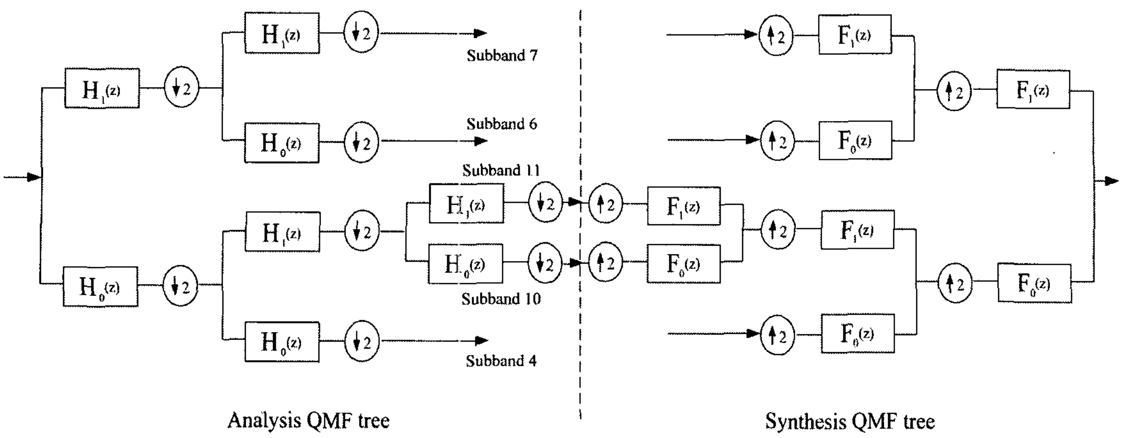
Soman et al, derived the coding gain for arbitrary subband decomposition, but they do not deal with linearly phased filter bases[1]. Katto et al, derived the gain for any kind of filter basis and for multilayer subband decompositions like wavelet tree decomposition[2], but their coding gain does not cover an arbitrary subband decomposition such as wavelet packet decomposition[3]. Katto also optimized the filters from the view of energy compaction, with the certain statistic model of input signal. Such trials to optimize filter coefficient under the statistic model of input signal was also made by Uzun et al, for two channel filter banks[4], and Caglar et al, for orthogonal filter banks[5]. However, the approach to construct the optimal subband decomposition has not been made, as far as the author's knowledge goes.

The remaining of this paper is organized as follows: In section 1, notations will be defined, and the wavelet packed based SBC will be analyized in connection with the notations. Section 2 derives the unified coding gain and defines the optimization problem mathematically. In section 3, a procedure to find the optimal wavelet packet structure based on dynamic programming will be presented. Applications of this proposed algorithm are reported in section 4. Finally, the conclusion is drawn in section 5.

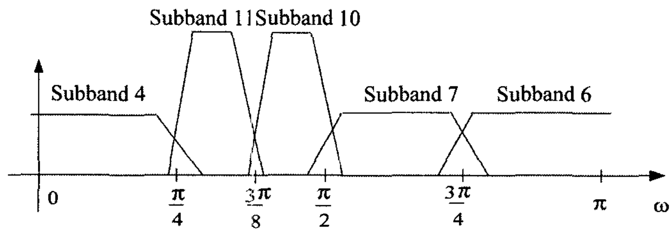
Ⅱ. Wavelet packet subband decomposition

The tree structured connections of QMFs realize a non-uniform subband decompositions, which implies that the QMF tree realizes the wavelet packet bases. An example of a tree structured QMF is shown in Figure 1, where $H_o(z)$, $H_1(z)$ re the transfer functions of low and high analysis filters and $F_o(z)$, $F_1(z)$ are the low and high synthesis filters[6]. If the non-uniform subband decomposition is a perfect reconstruction (PR) system, $H_o(z)=-F_1(z)$, $H_1(z)=F_o(-z)$. The QMF tree can be represented by a binary tree. For mathematical analysis, it is necessary to formulate the tree in association with filter bank theory as followings:

- S : The set of nodes of a QMF tree. The root node is numbered as 1. Node $2i$ and node $(2i+1)$ are child nodes of node i. A branch from node i to node $2i$ performs a lowpass filtering either by $H_o(z)$ followed downsampler in analysis filter banks, or by $F_o(z)$ following an upsampler in synthesis banks. A branch from node i to node $(2i+1)$ performs a highpass filtering by either $H_1(z)$ followed by a downsampler in analysis filter banks, or by $F_1(z)$ following an upsampler in synthesis banks. A subband number is labeled as the corresponding node number.
- $L(S)$: The set of leaf nodes of tree S. Subbands corresponding to leaf nodes determine a subband decomposition. We use $|L(S)|$ to denote the number of channels (i.e, the number of subbands).



(a) Tree-structured connections of QMFs



(b) Non-uniform subband decomposition by the above QMF tree

그림 1. 나무 구조 QMF와 그 비균일 서브밴드 분해의 예. 이 QMF 나무의 최종 매듭 집합은 L(S) = {4, 6, 7, 10, 11}이다

Fig 1. An example of tree structured QMFs and its nonuniform subband decomposition. The leaf node set of this QMF tree is L(S)={4,6,7,10,11}.

- d_i : Depth of node i, That is, $d_i = \lfloor \log_2 i \rfloor$, where $\lfloor x \rfloor$ is the greatest integer that is smaller than x. Thus, the subband size of the node i is $\pi/2^{d_i}$, and $\sum_{i \in L(S)} \frac{1}{2^{d_i}} = 1$.
- $h^i(n), H^i(z)$: The analysis filter, and its response which construct subband i. This filter bank is equivalent to the di connections of QMF banks.
- $f^i(n), F^i(n)$: The synthesis filter, and its response which are counter parts of $h^i(n), H^i(z)$.

Let the indexed-subscript ε_j be 0, or 1. $\varepsilon_j=0$ indicates a lowpass filter at the jth level of the QMF tree while $\varepsilon_j=1$ indicates a highpass filter. Then the analysis filter $h_{\varepsilon_1,\ldots,\varepsilon_d}(n)$ corresponding to the connection of QMFs along the path $\{\varepsilon_1, \ldots, \varepsilon_d\}$ is calculated by the following recursively performing convolutions [6].

$$h_{\varepsilon_1,\ldots,\varepsilon_d}(n) = h_{\varepsilon_d}(n) * h_{\varepsilon_1,\ldots,\varepsilon_{d-1}}(n). \tag{1}$$

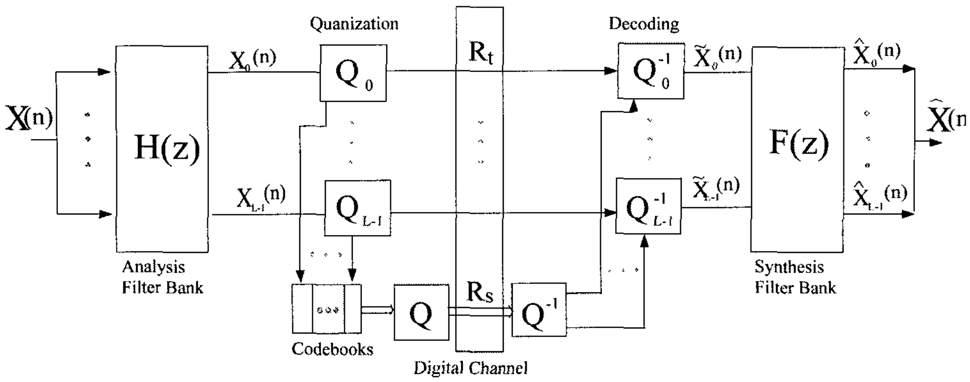
Here, the bandwidth of $h_{\varepsilon_1,\ldots,\varepsilon_d}(n)$ is $\pi/2^d$, and $h^i(n) = h_{\varepsilon_1,\ldots,\varepsilon_d}(n)$, if $i = 2^d + \sum_{j=1}^{d} \varepsilon_j \cdot 2^{d-j}$. Therefore, the signal filtered by the analysis bank is represented by

$$y_{\varepsilon_1,\ldots,\varepsilon_d}(n) = \sum_{m=-\infty}^{\infty} x(m) h_{\varepsilon_1,\ldots,\varepsilon_d}(2^d \cdot n - m) \tag{2}$$

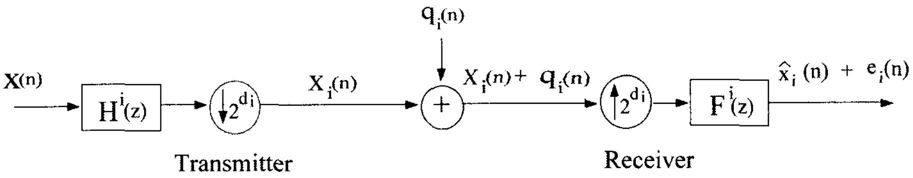
where $x(n)$ is an input signal. The corresponding synthesis filter $f_{\varepsilon_1,\ldots,\varepsilon_d}(n)$ can be also recursively defined as

$$f_{\varepsilon_1,\ldots,\varepsilon_d}(n) = \begin{cases} f_{\varepsilon_d}(n) * f_{\varepsilon_1,\ldots,\varepsilon_{d-1}}(\frac{n}{2}) & n\text{: even} \\ 0 & n\text{: odd} \end{cases} \tag{3}$$

So, the equivalent synthesized signal is



(a) SBC System



(b) A channel system based on quantization noise model

그림 2. (a) 서브밴드 부호화 시스템. (b) 서브밴드 크기가 $\pi/2^{d_i}$인 채널

Fig. 2. (a) Subband Coding System. (b) A channel whose subband size is $\pi/2^{d_i}$.

$$x_{\varepsilon_1,\ldots,\varepsilon_d}(n) = \sum_{m=-\infty}^{\infty} y(m)\cdot f_{\varepsilon_1,\ldots,\varepsilon_d}(n-2^d\cdot m) \quad (4)$$

where y(n) is input signal to the filter, $f_{\varepsilon_1,\ldots,\varepsilon_d}(n)$.

Ⅲ. Formal Problem Definition

1. Unified Coding Gain

Figure 2 depicts a practical SBC system and magnifies one of the channels. Without loss of generality, we can assume that the subband decomposition of the SBC is S, and the subband size of the depicted channel is $\pi/2^{d_i}$, $i\in L(S)$. In order to make theoretical analysis be possible, the quantization noise $q_i(n)$ embedded into the subband i is assumed to be an additive white process and uncorrelated with the subband signal $x_i(n)$. From this uncorrelation, the channel reconstruction error $e_i(n)$ is produced by only $q_i(n)$ and is also uncorrelated with the channel reconstruction signal $\hat{x}_i(n)$ and other channel reconstruction error $e_j(n)$, $i\neq j$. Therefore, the reconstructed signal of the SBC is $\hat{x}(n)=\sum_{i\in L(S)}\{\hat{x}_i(n)+e_i(n)\}=x(n)+e(n)$, where $x(n)$ is the original signal and e(n) is the reconstruction error of the SBC and $E\{|e(n)|^2\}=\sum_{i\in L(S)}E\{|e_i(n)|^2\}$.

Through the following analysis, the reconstruction error variance for a nonuniform subband coder is obtained in terms of the subband signal variances.

Therem 1 : The reconstruction error variance of an SBC $E\{|e(n)|^2\}$ satisfies following inequality:

$$E\{|e(n)|^2\} = \sum_{i\in L(S)}\frac{\sigma^2_{q_i}}{M_i}\sum_m f^i(m)| \ge \prod_{i\in L(S)}(\sigma^2_{q_i}\cdot\sum_m |f^i(m)|^2)^{1/M_i} \quad (5)$$

where $\sigma^2_{q_i}$ is the quantization noise variance of the subband i whose size is $\pi/M_i=\pi/2^{d_i}$, and $f^i(m)$ is the synthesis filter to construct the subband i.

Proof: The wide sense stationary (WSS) signal after an interpolator becomes a cyclostationary process with the period of the interpolation rate [7]. Therefore, the channel reconstruction error $e_i(n)$ in Figure 2 is a cyclostationary process with the period M_i. The periodic autocorrelation of $e_i(n)$ is obtained as following;

$$\begin{aligned} R_e(n, n+m) &= E\{e_i(n)e_i(n+m)\} \quad \text{from} \quad (4) \\ &= \sum_r\sum_s E\{q_i(r)\cdot q_i(s)\}\cdot f^i(n-rM_i)\cdot f^i(m+n-sM_i) \\ &= \sum_r\sum_s R_{qq}(r-s)\cdot f^i(n-rM_i)\cdot f^i(m+n-sM_i). \end{aligned}$$

And, the average autocorrelation of a cyclostationary process is computed by taking the time average with respect to cyclo-period [8]. Thus, we have

$$\begin{aligned} \overline{R_e}(n) &= \frac{1}{M_i}\sum_{l=0}^{M_i-1} R_e(l, n+l) \\ &= \frac{1}{M_i}\sum_{l=0}^{M_i-1}\sum_r\sum_s f^i(l-rM_i)\cdot f^i(l+n-sM_i)\cdot R_{qq}(r-s) \\ &= \frac{\sigma_{q_i}^2}{M_i}\sum_{l=0}^{M_i-1}\sum_r f^i(l-rM_i)f^i(l+n-rM_i). \end{aligned}$$

The middle equation above comes from $R_{qq}(r-s)=\sigma_{q_i}^2\cdot\delta(r-s)$, because the quantization noises are white processes.

From $E\{|e_i(n)|^2\}=\overline{R_e}(0)$, $E\{|e_i(n)|^2\}$ is calculated such that

$$E\{|e_i(n)|^2\} = \frac{\sigma_{q_i}^2}{M_i}\sum_{l=0}^{M_i-1}\sum_r |f^i(l-rM_i)|^2 = \frac{\sigma_{q_i}^2}{M_i}\sum_m |f^i(m)|^2.$$

Since $\sum_{i\in L(S)} 1/2^{d_i}=1$, the arithmetic and geometric mean inequality leads to following;

$$E\{|e(n)|^2\} = \sum_{i\in L(S)} E\{|e_i(n)|^2\} \geq \prod_{i\in L(S)}\left(\sigma_{q_i}^2\cdot\sum_m |f^i(m)|^2\right)^{1/d_i}.$$

Therefore, (5) is proved. Q.E.D.

The filter energy $\sum_m |f^i(m)|^2$ represents the factor of the distribution of energy by the subband filter and may be regarded as a weight factor for the subband. This weight factor takes into account the quantization noise leakage to other channels. For orthonormal bases, $\sum_m |f^i(m)|^2=1$, so the quantization noise does not suffer from energy leakage to other channels.

Following the usual way to derive the conventional coding gain[9],[10], we obtain the unified coding gain for a subband decomposition S as follow:

$$G(S) = \frac{\sigma_x^2}{\prod_{i\in L(S)}\left(\beta_i\sigma_{x_i}^2\right)^{1/2^{d_i}}} \tag{6}$$

where $\beta_i=\sum_m |f^i(m)|^2$, σ_x^2 is the signal variance of input signal, and $\sigma_{x_i}^2$ is the signal variance of subband i. From (2) the subband signal variances are expressed in terms of the input signal correlation $R_{xx}(k)$ such as

$$\begin{aligned} \sigma_{x_i}^2 &= E\{|x_i(n)|^2\} \\ &= \sum_m\sum_l h^i(2^{d_i}n-m)\cdot h^i(2^{d_i}n-l)\cdot R_{xx}(m-l) \end{aligned} \tag{7}$$

Therefore, if the input signal correlation is known, the filter coefficients can be also optimized in sense of the coding gain for a fixed subband decomposition, after plugging (7) into (6).

2 Problem Definition

As seen in (6), the geometric combination of the subband signal variances decides the coding gain and the combination is determined by subband decompositions. Therefore, the task of this study is to construct the optimal subband decomposition yielding the maximum coding gain for a given complexity. Generally, the coding gain improves as the number of subbands increases. But, the larger number of subbands requires the more implementation complexity which is limited in real world. Let $W(S)$ be the implementation cost distribution, that is, $W(S)=\{w_i \mid i\in L(S)\}$, where w_i is the cost for implementing the subband i. Some examples of w_i are the number of filter, the computation load, or the number of subbands. For the number of filter banks, $w_i = d_i$ because d_i branches are needed from the root node to node i, and each branch contains one filter. For the computation load, $w_i = \sum_{k=1}^{d_i} 1/2^k$ since the subband signal size at each node is directly proportional to the subband size at the node. For the number of subbands, $w_{i=1}$ since a leaf node generates one subband. The total implementation complexity is denoted as $|W(S)| = \sum_{i \in L(S)} w_i$.

For a given implementation complexity $W(S)$, the problem is formally defined as;

$$\min_S\ G(S) = \frac{\sigma_x^2}{\prod_{i \in L(S)} (\beta_i \sigma_{x_i}^2)^{1/2^{d_i}}} \tag{8}$$

$$\text{subject to} \quad |W(S)| = \sum_{i \in L(S)} w_i \le W$$

After taking the logarithm through (6), the optimization problem (8) can be restated as

$$\min_S\ O(S) = O(S) = \min_S \left\{ \sum_{i \in L(S)} \frac{1}{2^{d_i}} \log(\beta_i \sigma_{x_i}^2) \right\} \tag{9}$$

$$\text{subject to} \quad |W(S)| = \sum_{i \in L(S)} w_i \le W.$$

Ⅳ. Optimization Algorithm

A direct way to solve the above optimization problem (9) is to search all possible subband decompositions for the optimal subband structure that bears the smallest lower distortion bound. Since such an exhaustive does not prune the non-optimal subtrees, the expensive computation loads required to carry it out are not justified. Furthermore, to construct possible subband decompositions according to all kinds of implementation cost would be a mammoth task. For example, there exist $2^{(N-1)}$ subband decompositions for N subbands, and the subband number NS is an irregular function of cost $w_i, i \in L(S)$. Therefore, a fast pruning method is required instead. This study uses an unconstrained optimization problem with Lagrange multiplier to solves (9).

1. Performance Analysis as Number of Subbands

In general, the reconstruction error of an SBC decreases as number of subbands increases. In order to understand the better performance of the higher subband numbers, one must observe the change in the reconstruction error variance when subbands are increased by one. If a two-channel QMF splits a subband i into two disjoint half subbands $2i$, $2i+1$ each of which covers disjoint half of the parent subband, the subbands are increased by one. It can be deduced from (9) that the portion of reconstruction error variance due to the signal of subband i is $(1/2^{d_i}) \log \beta_i \sigma_{x_i}^2$. In the same way, the reconstruction error variance due to the signals of subband $2i$, $2i+1$ is

$$\frac{1}{2^{d_{2i}}} \log \beta_{2i} \sigma_{x_{2i}}^2 + \frac{1}{2^{d_{2i+1}}} \log \beta_{2i+1} \sigma_{x_{2i+1}}^2 = \frac{1}{2^{d_i}} \log (\beta_{2i} \sigma_{x_{2i}}^2 \beta_{2i+1} \sigma_{x_{2i+1}}^2)^{1/2}$$

since $d_{2i} = d_{2i+1} = d_i + 1$. Therefore, the change in the reconstruction error variance caused by splitting a subband to increase the subband number is as follow

$$\Delta_i = \frac{1}{2^{d_i}} \log (\beta_{2i} \sigma_{x_{2i}}^2 \beta_{2i+1} \sigma_{x_{2i+1}}^2)^{1/2} - \frac{1}{2^{d_i}} \log \beta_i \sigma_{x_i}^2 = \frac{1}{2^{d_i}} \log \left\{ \frac{(\beta_{2i} \sigma_{x_{2i}}^2 \beta_{2i+1} \sigma_{x_{2i+1}}^2)^{1/2}}{\beta_i \sigma_{x_i}^2} \right\}$$

For most of practical filter banks constructed by wavelet basis, the weight factors β_i, β_{2i} and β_{2i+1} are sufficiently close to 1 when compared to the signal variances $\sigma_{x_i}^2$, $\sigma_{x_{2i}}^2$ and $\sigma_{x_{2i+1}}^2$. So, Δ_i can be approximated as

$$\Delta_i \simeq \frac{1}{2^{d_i}} \log \left\{ \frac{(\sigma_{x_{2i}}^2 \sigma_{x_{2i+1}}^2)^{1/2}}{\sigma_{x_i}^2} \right\} \tag{10}$$

Because the numerator term of Δ_i forms the geometric mean of $\sigma^2_{x_{2i}}$, $\sigma^2_{x_{2i+1}}$, Δ_i reaches to the maximum when $\sigma^2_{x_{2i}}$ and $\sigma^2_{x_{2i+1}}$ are getting closer. Recalling that the subband i is split by the lowpass filter $H_0(z)$ to generate the subband $2i$ and the highpass filter $H_1(z)$ to generate the subband $2i+1$, $\sigma^2_{x_{2i}}$ and $\sigma^2_{x_{2i+1}}$ are closest when $S_{x_ix_i}(e^{j\omega})$ is constant. The constant $S_{x_ix_i}(e^{j\omega})=\sigma^2_{x_i}$ is possible, only when $x_i(n)$ is a white process. The power spectrums of $x_{2i}(n), x_{2i+1}(n)$ are expressed in aliasing terms of the power spectrum of $x_i(n)$, following[6];

$$S_{x_kx_k}(e^{j\omega})=\frac{1}{2}\sum_{k=0}^{1}|H_l(e^{j(\omega-2\pi k)/2})|^2 S_{x_ix_i}(e^{j(\omega-2\pi k)/2}) \quad (11)$$

where $H_l(z)=H_0(z)$ if $k=2i$, and $H_l(z)=H_1(z)$ if $k=2i+1$. Applying $S_{x_ix_i}(e^{j\omega})=\sigma^2_{x_i}$ and the Parseval theorem to (11) obtains $\sigma^2_{x_{2i}}=\sigma^2_{x_i}\sum|h_0(m)|^2$, and $\sigma^2_{x_{2i+1}}=\sigma^2_{x_i}\sum|h_1(m)|^2$. Therefore, the maximum of Δ_i will be

$$\Delta_{\max}\simeq\frac{1}{2^{d_i}}\log\left(\sum|h_0(m)|^2\sum|h_1(m)|^2\right)^{1/2}.$$

The filter energies $\sum|h_0(m)|^2, \sum|h_1(m)|^2$ are also approximately 1 for practical filters, so $\Delta_{\max}\simeq 0$. Furthermore, in meaningful signals, such as an image signal and a speech signal, their energies are concentrated on certain frequency bands, which implies that $\Delta_i \ll \Delta_{\max}$. Therefore, in practical cases, the reconstruction error variance change due to splitting a subband i will be

$$\Delta_i=\frac{1}{2^{d_i}}\log\left\{\frac{|\beta_{2i}\sigma^2_{x_{2i}}\,\beta_{(2i+1)}\sigma^2_{x_{(2i+1)}}|^{1/2}}{\beta_i\sigma^2_{x_i}}\right\}\ll\Delta_{\max}\simeq 0. \quad (12)$$

The conclusion is that Δ_i is negative, and that splitting subbands decreases the reconstruction error variance of an SBC.

Lemma 1: If splitting a subband satisfies (12), the reconstruction error variance function $O(S)=\sum_{i\in L(S)}\frac{1}{2^{d_i}}\log(\beta_i\sigma^2_{x_i})$ is a monotonically decreasing function of the number of subbands.

Proof: We prove this lemma by the following induction on the subband number $|L(S)|$. Let S_n be the subband decomposition having n subbands, that is, $|L(S_n)|=n$. The subband S_n is generated from S_{n-1} by splitting a subband $i\in L(S_{n-1})$. From (12), $|L(S_2)|=2$. Assume that it is also ture for $|L(S_n)|=n$. Now, the node $i\in L(S_n)$ splits into the two child nodes $2i$, $2i+1$. The split structure's subband number is $|L(S_{n+1})|=n+1$. Let $L^*(S_n)=L(S_n)-\{i\}$, and $L^*(S_{n+1})=L(S_{n+1})-\{2i, 2i+1\}$. Then, $L^*(S_n)=L^*(S_{n+1})$, and $|L^*(S_n)|=n-1$. Therefore,

$$\begin{aligned}O(S_{n+1}) &= \sum_{j\in L(S_{n+1})}\frac{1}{2^{d_j}}\log\beta_j\sigma^2_{x_j}\\ &= \sum_{j\in L^*(S_{n+1})}\frac{1}{2^{d_j}}\log\beta_j\sigma^2_{x_j} + \sum_{j\in\{2i,2i+1\}}\frac{1}{2^{d_j}}\log\beta_j\sigma^2_{x_j}\\ &< \sum_{j\in L^*(S_n)}\frac{1}{2^{d_j}}\log\beta_j\sigma^2_{x_j}+\frac{1}{2^{d_i}}\log\beta_i\,{\sigma_{x_i}}^2\end{aligned}$$

$$\text{from (12)} = \sum_{j\in L(S_n)}\frac{1}{2^{d_j}}\log\sigma^2_{x_j} = O(S_n).$$

Hence, $O(S_{n+1})<O(S_n)$. This induction proves the lemma. Q.E.D.

2. Dynamic Program for Optimal Subband Decomposition

The next step is that the constrained optimization problem (9) is converted into an unconstrained optimization problem with Lagrange multiplier. The unconstrained problem develops a fast dynamic program that prunes non-optimal sub-decompositions appearing as non-optimal sub-trees.

Theorem 2: If the implementation complexity function $|W(S)|$ is increasing for the subband number $|L(S)|$, then the unconstrained problem for fixed $\lambda>0$,

$$\min_S \{ O(S) + \lambda |W(S)| \}$$
$$= \min_S \left\{ \sum_{i \in L(S)} \frac{1}{2^{d_i}} \log \sigma^2_{x_i} + \lambda \cdot \sum_{i \in L(S)} w_i \right\} \quad (13)$$

solves the constrained optimization problem (9). The optimal subband decomposition S^* occurs at $|W(S^*)| = W_c$.

Proof: If the implementation complexity $|W(S)|$ is increasing for the subband number $|L(S)|$, it follows from Lemma 1 that the distortion function $O(S)$ is strictly decreasing for $|W(S)|$. Let S^* the solution of (13). Then for any subband structures satisfying $W(S) \le W_c$, we have

$$O(S^*) + \lambda \cdot |W(S^*)| \le O(S) + \lambda \cdot |W(S)|$$

or equivalently

$$O(S^*) - O(S) \le \lambda \cdot (|W(S)| - |W(S^*)|)$$

and so $O(S^*) - O(S) \le \lambda \cdot (|W(S)| - W_c)$

which is equivalent to

$$O(S^*) \le O(S) \quad \text{for} \quad |W(S)| \ge |W(S^*)| .$$

Because $O(S)$ is strictly decreasing for $|W(S)|$ and $\lambda > 0$, the unconstrained problem of (13) is identical to the constrained problem of (9), and the solution S^* occurs at $|W(S^*)| = W_c$. Q.E.D.

Theorem 2 implies that as λ sweeps over positive numbers, all the operating points of implementation cost and distortion $(O(\lambda), W(\lambda))$ are created and draw a convex hull.

Regarding the cost function $W(S)$ as a rate function, the convex hull is equivalent to the

표 1. 최적 서브밴드 분해를 위한 알고리즘
Table 1. Algorithm for Optimal Subband Decomposition.

ALGORITHM Optimal Subband Decomposition
- Preset Maximum dept d and implementation complexity limit W_c.
- Store node values $n_i = \frac{1}{2^{d_i}} \log \sigma^2_{x_i}$, w_i at every node.
- $L(S) \leftarrow \{ 2^d, \cdots, (2^{d+1}-1) \}$.
- Set $\lambda' > \lambda''$, such that $W(\lambda') < W_c < W(\lambda'')$.

● step 1): /*Construct an optimal tree for $\lambda_{op} > 0$.*/
$\lambda_{op} \leftarrow |O(\lambda') - O(\lambda'')| / |W(\lambda') - W(\lambda'')|$
  For $l = d$ to $l = 2$ /* for nodes at depth l */
    For $i = 2^{l-1}$ to $i = 2^l - 1$ /* for nodes i at depth l */
      /* If(parent node's Lagrangian Cost > sum of child node's Lagrangian Costs) then split the node i. Otherwise prune the node i. */
      If $(n_i + \lambda w_i) > (n_{2i} + \lambda w_{2i}) + (n_{2i+1} + \lambda w_{2i+1})$, /* Split node i */
        then $n_i \leftarrow n_{2i} + n_{2i+1}$, $w_i \leftarrow w_{2i} + w_{2i+1}$,
      Else /* Prune the subtree hang on node i */
        then $L(S) \leftarrow L(S) - \{2i, 2i+1\} + \{i\}$.
      $i \leftarrow i+2$. /* for i */
      $l \leftarrow l-1$. /* for l */

● step 2): /* Update λ' and λ''. */
If $(W(\lambda_{op}) > W_c)$, $\lambda' \leftarrow \lambda''$. Go to step 1).
Elseif $(W(\lambda_{op}) < W_c)$, $\lambda'' \leftarrow \lambda_{op}$.
Elseif $(W(\lambda_{op}) = W_c)$, stop.

typical rate-distortion (R,D) curves. The difference between (O,W) curve and (R,D) curve lies in their different cost functions. W treats implementation complexity as proportional to the number of subbands, whereas R is a coding bit rate. Therefore, the proposed algorithm substitutes the implementation cost function $W(S)$ for the bit rate R as found in conventional (R, D) function. This make it possible to apply the fast dynamic programming technique appeared in[3],[11]. The essence of the algorithm is that at each node the Lagrangian costs of parent node and child nodes are compared; if the Lagrangian cost by child nodes is more expensive, then the sub-tree hanging on the parent node is pruned. The tree formed by the surviving paths is optimal for a fixed $\lambda > 0$, therefore, the algorithm constructs only the optimal tree for a given $\lambda > 0$. The algorithm is shown in Table I.

V. Applications

To demonstrate the proposed algorithm, the optimal subband decomposition for Chirp signal is constructed. The implementation complexity is set up as the number of subbands, so $w_i = 1, i \in L(S)$ and $|W(S)| = |L(S)|$. The implementation complexity budget is $W_c = 3$. The QMF filter adopts the biorthogonal linearly-phased spline wavelet filter that is popularly used for image compression[12]. The filter coefficients are listed in Table 2. The subband weight factors β_i are calculated from (3) and shown in Table 3. Figure 3

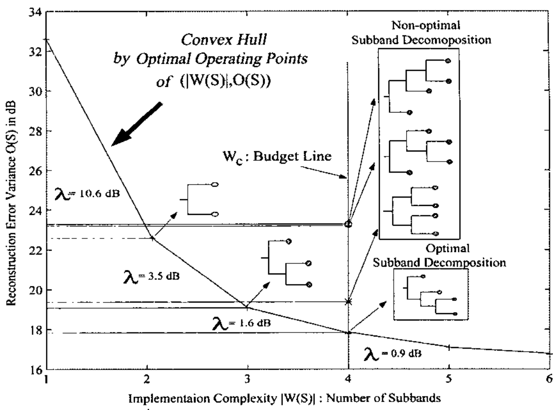


그림 3. Chirp 신호에 대한 $(W(S), O(S))$ 곡선과 그 최적 서브밴드 분해. 위쪽의 분기는 고주파 통과 필터를 나타내고 아래쪽의 분기는 저주파 통과 필터를 나타낸다

Fig. 3. The $(W(S), O(S))$ curve for a Chirp signal and its optimal subband decompositions. An upper branch represents high-pass filtering and a lower branch represents low-pass filtering.

depicts the Chirp signal and its subband signals. Figure 4 presents $(G(S), W(S))$ curve and the optimal subband decompositions at each implementation complexities. The optimal subband decompositions occur at each convex hull points. From Figure 4, the optimal subband decomposition for $W_c = 3$ is $L(S)$={3, 4, 5}.

With drastically reducing computation complexities, another advantage of the proposed

표 2. 9-7 이중직교 웨이블릿 필터

Table 2. 9-7 Biorthogonal Wavelet Filter.

n	0	±1	±2	±3	±4
$2^{-1/2} h_0(n)$	0.602949	0.266864	-0.078223	-0.016864	0.026749
$2^{-1/2} f_0(n)$	0.557543	0.295636	-0.028772	-0.045636	

표 3. 서브밴드 가중 인자 β_i

Table 3. Subband weight factor β_i.

subband i	2	3	4	5	6	7	8	9	.....
β_i	0.98	1.04	0.97	1.02	1.02	1.08	0.95	1.07	....

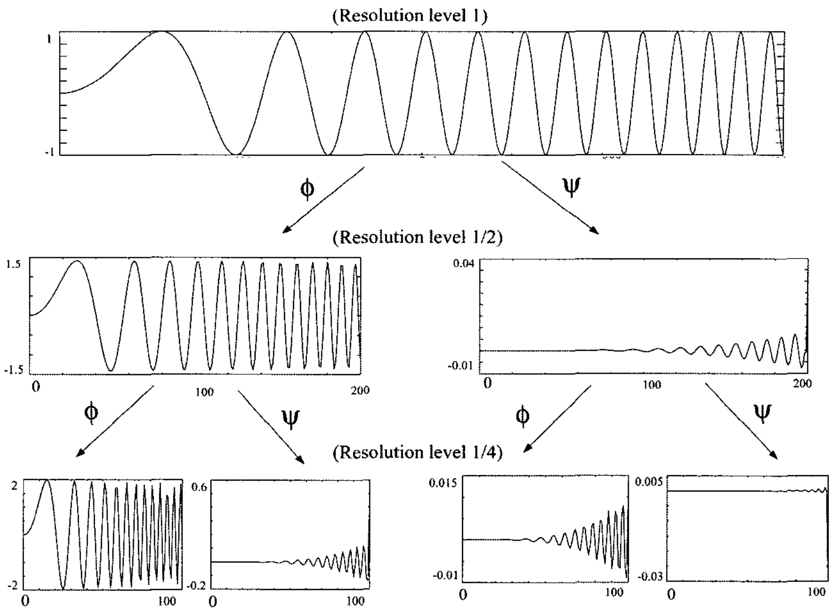


그림 4. Chirp 신호와 그 서브밴드 신호들. ϕ는 스케일링 필터, 즉 저주파 통과 필터이고 Ψ는 웨이블릿 필터, 즉 고주파 통과 필터이다

Fig. 4. Chirp signal and its subband signals. ϕ is the scaling filtering, i.e., the low-pass filtering and Ψ is the wavelet filtering, i.e., the high-pass filtering.

algorithm is that it discovers the reasonable implementation complexity. As Figure 4 indicates, for the Chirp signal, 4 or 5 subbands can reach saturation in the coding gain improvement, which means that such numbers of subbands are reasonable implementation complexities.

Ⅵ. Conclusions

In this paper, the unified coding gain which works for any kinds of FIR filter basis and arbitrary binary subband decompositions was derived. This paper also developed a fast pruning algorithm to construct the binary subband decomposition producing the maximum coding gain for a given limit of implementation complexity. The algorithm not only drastically reduces computational complexities but also discovers the reasonable implementation complexity for a given signal.

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