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The MLE and the UMVUE of the Right-Tail Probability in a Levy Distribution

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Abstract

MLE and UMVUE of the right-tail probability in a Levy distribution will be considered, and hence the UMVUE is more efficient in a sense of simulated MSE than the MLE of the right-tail probability.

Key Words and Phrases: MLE, UMVUE, The right-tail probability, Levy distribution, MSE

1. Introduction

We shall consider the Levy distribution with the following pdf:

$$f(x;\sigma) = \sqrt{\frac{\sigma}{2\pi}} x^{-3/2} \exp(-\frac{\sigma}{2x}), \quad x > 0, \quad \text{where } \sigma > 0. \tag{1.1}$$

(see O'Reilly & Rueda(1998)).

This Levy distribution is a special case of the inverted gamma distribution with the shape 1/2 and scale $2/\sigma$ (see O'Reilly & Rueda(1998)). An inverse Gaussian density with parameters (λ, μ) converges to the Levy density as $\theta \to 0$ when $\theta = \lambda/\mu$ and $\sigma = \lambda$ (see O'Reilly & Rueda(1998). The Levy random variable doesn't have moments of all orders, but it has been very useful in applications, analysis of stock prices and physics.

The Levy distribution has been used in physics by Montroll & Shlesinger(1983), whereas Jurlewicz & Weron(1993) considered the distribution.

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Here we shall consider the MLE and the UMVUE of the right tail probability in a Levy distribution, and compare the simulated MSE's of the MLE and the UMVUE of the right-tail probability in a Levy distribution, because the second moments of the MLE and the UMVUE can't be obtained by closed forms exactly.

2. MLE of the right-tail probability

The right-tail probability of the Levy random variable with pdf (1.1) is

$$\text{for } t > 0, \quad R(t) = \frac{1}{\sqrt{\pi}} \cdot \gamma(\frac{1}{2}, \frac{\sigma}{2t}), \\ = erf(\sqrt{\frac{\sigma}{2t}}), \tag{2.1}$$

where $\gamma(a, b)$ and $erf(x)$ are the incomplete gamma function and the error function, respectively. The first and second equalities of (2.1) come from the formula 3.381(1) and the formula 8.359(4) in Gradshteyn & Ryzhik(1965), respectively.

Assume $X_1, X_2, \cdots, X_n$ is a simple random sample from the Levy distribution with pdf (1.1). Then the MLE of σ is

$$\hat{\sigma} = \frac{n}{\sum_{i=1}^{n} \frac{1}{X_i}} \tag{2.2}$$

By the factorization Theorem in Rohatgi(1976), we can obtain the followings easily: Fact 1. $S = \sum_{i=1}^{n} \frac{1}{X_i}$ is a complete sufficient statistic of σ, and follows a gamma distribution with a shape parameter $n/2$ and a scale parameter $2/\sigma$.

From the results (2.1) & (2.2), the MLE of the right-tail probability in the Levy distribution is given by

$$\widehat{R(t)} = erf(\sqrt{\frac{n}{2t \cdot S}}), \quad S = \sum_{i=1}^{n} \frac{1}{X_i} \tag{2.3}$$

Since some regular conditions are satisfied, by Remark 4 in p.362 of Rohatgi(1976) and the result (2.1), the Cramer-Rao lower bound(CRLB) for variance of an unbiased estimator of R(t) can be obtained by

$$\text{CRLB}(\sigma) = \frac{1}{n \cdot \pi} \cdot \frac{\sigma}{t} \cdot e^{-\frac{\sigma}{t}} \tag{2.4}$$

By the large sample property, an asymptotic properties of MLEs(p.316) in Bain(1992), an asymptotic$(1-\alpha)100\%$ confidence interval of $\widehat{R(t)}$ using the ML estimator can be obtained as

$$(\widehat{R(t)} - z_{\alpha/2}\cdot\sqrt{CRLB(\hat{\sigma})}, \widehat{R(t)} + z_{\alpha/2}\cdot\sqrt{CRLB(\hat{\sigma})}). \qquad (2.5)$$

By the formula 3.471(9) in Gradshteyn & Ryzhik(1965), expectation of its confidence length can be obtained as

$$\frac{z_{\alpha/2}}{\Gamma(\frac{n}{2})\sqrt{n\cdot 2^{n-4}\pi}}(\frac{n\sigma}{t})^{\frac{n+1}{4}}K_{\frac{n-1}{2}}(\sqrt{\frac{n\sigma}{t}}), \qquad (2.6)$$

where $K_r(x)$ is the modified Bessel function of order r.

From the MLE of $R(t)$, its expectation can be obtained by mathematical special functions: From definition of an error function and the result (2.3),

$$\widehat{R(t)} = \sqrt{\frac{2n}{\pi t}}\int_0^1 S^{-1/2}e^{-\frac{ny^2}{2tS}}dy,$$

and hence, from Fact 1 we can obtain the following:

$$\begin{aligned} E(\widehat{R(t)}) &= \sqrt{\frac{2n}{\pi t}}\frac{1}{\Gamma(n/2)(2/\sigma)^{n/2}}\int_0^1 dy\int_0^\infty s^{n/2-3/2}e^{-\frac{\sigma}{2}s-\frac{ny^2}{2ts}}ds \\ &= \sqrt{\frac{2n}{\pi t}}\frac{1}{\Gamma(n/2)(2/\sigma)^{n/2}}\int_0^1 2(\frac{n}{\sigma t}y^2)^{(n-1)/4}K_{\frac{n-1}{2}}(\sqrt{\frac{n\sigma}{t}}\cdot y)dy \\ &= (\frac{n\sigma}{t})^{-(n-3)/4}\cdot[K_{(n-1)/2}(\sqrt{n\sigma/t})\cdot L_{(n-3)/2}(\sqrt{n\sigma/t}) \\ &\qquad + K_{(n-3)/2}(\sqrt{n\sigma/t})\cdot L_{(n-1)/2}(\sqrt{n\sigma/t})]. \end{aligned} \qquad (2.7)$$

where $L_r(x)$ is the modified Struve function(8.55 in Gradshteyn & Ryzhik(1965)).

The second and the third equalities in (2.7) come from the formula 3.471(9) and 6.561(4) in Gradshteyn & Ryzhik(1965), respectively.

3. UMVUE of the right-tail probability

Assume $X_1, X_2, \cdots, X_n$ is a simple random sample from the Levy distribution with pdf (1.1). Then, $U(X_1) \equiv \begin{cases} 1, & \text{if } X_1 \ge t \\ 0, & else \end{cases}$ is an unbiased estimator of $R(t) = P(X \ge t)$.

Since $S = \sum_{i=1}^n \frac{1}{X_i}$ is a complete sufficient statistic of σ, based on Lehmann-Scheffe Theorem in Rohatgi(1976), $E(U(X_1)|S = \sum_{i=1}^n \frac{1}{X_i})$ is a UMVUE of $R(t) = P(X \ge t)$.

To find the conditional expectation $E(U(X_1)|S = \sum_{i=1}^{n} \frac{1}{X_i})$, first we must find the conditional density $f(x|s)$ of X_1 given $S = s$:

$$\begin{aligned} f(x|s) &= \frac{f(s|x) \cdot f_{X_1}(x)}{f_S(s)} \\ &= \frac{1}{B(\frac{1}{2}, \frac{n-1}{2})} s^{-n/2+1} x^{-3/2} (s - \frac{1}{x})^{\frac{(n-1)}{2}-1}, \text{ if } x > 1/s \end{aligned} \quad (3.1)$$

where B(a,b) is the Beta function.

Therefore, using the result (4.1) and Fact 1,

$$\tilde{R}_U(t) = E(U(X_1)|S = \sum_{i=1}^{n} \frac{1}{X_i}) = \frac{1}{B(\frac{1}{2}, \frac{n-1}{2})} \cdot B(\frac{1}{2}, \frac{n-1}{2}; \frac{1}{tS}) \quad (3.2)$$

where B(a,b;x) is the incomplete Beta function and $S = \sum_{i=1}^{n} \frac{1}{X_i}$, is the UMVUE of the right-tail probability in the Levy distribution with the pdf (1.1).

It's very hard for us to represent the second moments of the MLE and the UMVUE of the right-tail probability in a Levy distribution as closed forms, so using the MLE $\widehat{R(t)}$ and UMVUE $\tilde{R}_U(t)$ of R(t), Table 1 shows that their simulated MSE's will be evaluated when $n = 10(10)50$, $\sigma = 1$ and true values $R(t) =$ 0.01, 0.05, 0.1, 0.2, 0.3, and 0.4. Through Table 1, the UMVUE of $R(t)$ is more efficient in a sense of simulated MSE than the MLE of $R(t)$ when $n = 10(10)50$, $\sigma = 1$ and true values $R(t) = 0.01,\ 0.05,\ 0.1,\ 0.2,\ 0.3$, and 0.4 And Cramer-Rao lower bound of variance for an unbiased estimator of $R(t)$ is smaller than MSE's of the MLE and the UMVUE of $R(t)$.

Table 1. The simulated MSE's of the MLE and the UMVUE of $R(t)$

in a Levy distribution when $\sigma = 1$(units are 10^{-4})

R(t)	Sample size	10	20	30	40	50
	MLE	0.08	0.03	0.02	0.02	0.01
0.01	UMVUE	0.07	0.03	0.02	0.01	0.01
	CRLB	0.05	0.03	0.02	0.01	0.01
	MLE	2.09	0.81	0.52	0.38	0.29
0.05	UMVUE	1.65	0.72	0.47	0.35	0.27
	CRLB	1.25	0.62	0.42	0.31	0.25
	MLE	8.22	3.20	2.05	1.49	1.13
0.1	UMVUE	6.53	2.84	1.88	1.38	1.07
	CRLB	4.95	2.48	1.65	1.24	0.99
	MLE	30.78	12.20	7.87	5.71	4.36
0.2	UMVUE	25.09	10.93	7.26	5.33	4.12
	CRLB	19.17	9.59	6.39	4.79	3.83
	MLE	61.90	25.28	16.45	11.98	9.18
0.3	UMVUE	52.61	23.06	15.34	11.27	8.73
	CRLB	40.76	20.38	13.59	10.19	8.15
	MLE	93.54	39.79	26.18	19.15	14.75
0.4	UMVUE	84.11	37.20	24.81	18.25	14.17
	CRLB	66.52	33.26	22.17	16.63	13.30

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