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A New Methodology for the Rapid Calculation of System Reliability of Complex Structures

Sooyong Park
Korea Institute of Construction Technology, Go-Yang, Korea

Abstract

It is quite difficult to calculate the collapse probability of a system such as statically indeterminate structure that has many possible modes or paths to complete failure and the problem has remained essentially unsolved. A structure is synthesized by several components or elements and its capacity to resist the given loads is a function of the capacity of the individual element. Thus it is reasonable to assess the probability of failure of the system based upon those of its elements. This paper proposes an efficient technique to directly assess the reliability of a complex structural system from the reliabilities of its components or elements. The theory for the calculation of the probability of a structural system is presented. The target requirements of the method and the fundamental assumptions governing the method are clearly stated. A portal frame and two trusses are selected to demonstrate the efficiency of the method by comparing the results obtained from the proposed method to those from the existing methods in the literature.

Keywords: system failure probability, reliability of components, structural system, statically indeterminate structure

1. INTRODUCTION

Recent developments in structural mechanics have shown that there is no rational explanation of the degree of absolute safety provided by structures designed using traditional working stress design or ultimate strength methods (Freudenthal et al. 1966; Task Committee on Structural Safety 1972). In response to these developments, the modern trend has been to take into consideration the random nature of the loads to which structures are subjected, and the variation in the material properties of the structural constituents. In other words, the loads impacting a structure, and the resistance of that structure are considered to be random variables. The safety margin provided by the structure is the amount by which the random resistance of the structure exceeds the random load applied to the structure. Failure is said to occur when the safety margin is less than, or equal to, zero. The relative safety of a structure is now expressed in terms of a probability of failure, which is the complement of the reliability of the structure.

Many attempts have been made on the evaluation of reliability of structures. In general, real structures which are highly indeterminate, in other words "complex", may fail in one or combination of several modes. The approaches to evaluation of structural safety such complex structures can be categorized into three methods: (a) numerical integration method; (b) failure path method; and (c) simulation.

In the numerical integration method, a failure mode is defined by a limit state surface which divides the space of basic variables into two sets, the safety set and the failure set (Ditlevsen 1979). The probability of failure can be calculated by integration of the probability mass density functions over the failure set. Some researchers have attempted to evaluate the probability of failure by integrating numerically the joint probability density function of the applied loads over the failure set (Kam et al. 1983, Lin and Corotis; 1985). However, these approaches were limited to two-dimensional spaces since direct integration is impractical in multi-dimensional space.

In the failure path method, reliability analysis of a structural system can be divided into two steps: (a) identification of failure modes; and (b) estimation of failure probabilities of individual modes and the overall system. The failure modes can be identified using the standard structural analysis and may be generated automatically as demonstrated by Watwood (1979) and Gorman (1981). However, real structures can have many different failure modes and it is infeasible to enumerate all the possible failure modes. Many studies have focused on the possibility of using dominant failure modes based on the assumption that safety of a structural system can be estimated efficiently using these dominant failure modes (Moses and Stahl 1978; Ang and Ma 1981; Murotsu et al. 1984; Thoft-Christensen and Murotsu 1986; Ranganathan and Deshpande 1987; Tung and Kiremidjian 1992). The last step of reliability analysis is calculation of failure probability of a system using combination of failure modes and bound techniques.

Monte-Carlo simulation has been often used to estimate the probability of failure and to verify the results of other reliability analysis methods (Ang and Tang 1984; Grimmelt and Schueller 1982; Schueller and Stix 1987; Ranganathan and Deshpande 1987). In this technique, the random loads and random resistance of a structure are simulated and these simulated data are used to find out if the structure fails or not, according to pre-determined limit states. The probability of failure is the relative ratio between the number of failure occurrence and the total number of simulations. This technique is easy to apply but when encountered low failure probability which is typical in real structural systems, the number of simulation is larger so the method is generally not practical for most realistic problems.

Although these previous works represent great strides in structural safety evaluation, many problems remain to be solved. One of obstacles in the area of reliability analysis is difficulty to calculate failure probability of complex structures, which are, for example, highly indeterminate and have many element, using currently developed methods. The objective of this paper is to present a practical methodology that can be used to estimate the system reliability of arbitrary structures and to provide evidence to support the validity of the methodology.

2. THEORY OF THE METHOD

The reliability or safety of a structure is its ability to withstand the design loads for a specified period of time. If the probability of failure of the system is given by P_{system}, the reliability of the system, R, is defined as

$$R = 1 - P_{system} \tag{1}$$

A practical methodology is presented below for the evaluation of the reliability of arbitrary structural systems. To meet this objective, the following four tasks will be accomplished: (a) describe the background of the methodology; (b) discuss a set of realistic target requirements for the methodology; (c) postulate a set of assumptions to be used as a basis for the methodology and describe the basic components of the methodology that are consistent with the target requirements and the basic assumptions; and (d) select a model used in this study to estimate the failure probabilities of components/elements.

(1) Background

A mechanical system is any combination of individual elements that are synthesized to perform a dedicated mechanical function. For the purposes of this study, any mechanical system may be assigned to one of the following three categories: series systems (See Figure 1), parallel systems (See Figure 2), or hybrid systems (See Figure 3). Series systems are those systems in which the failure of any element leads to the failure of the system. Parallel systems are those systems in which the combined failure of each and every element of the system results in the failure of the system. If a system does not satisfy these strict definitions of "series" or "parallel" systems, the system is classified as a hybrid system. Hybrid systems are those systems in which "few", "some", or "most" (but not "anyone" or "all") of the elements must fail in order for the system to fail. Any combination of elements whose failure results in the failure of the system is known as a failure mode. In the language of structural mechanics, hybrid systems are statically indeterminate (redundant) structures with many possible failure modes.

A hybrid system may be described as a series system of parallel modes of failure. For a single mode of failure in a system, the possibility exists for there to be statistical correlation between the demand on the elements in that mode or the capacity of the elements to resist the demand. Furthermore, since the same element may participate in more than one mode of failure, statistical correlation must also exist among the various failure modes of a hybrid (or series or parallel) system. In this study, hybrid systems in which statistical correlation exists among elements and failure modes are designated here to be *complex systems*. A major problem in reliability theory is the efficient and accurate determination of the failure probabilities of highly redundant structural systems with correlated modes of failure.

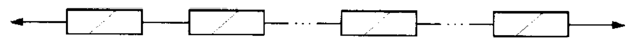


Figure 1. Series System

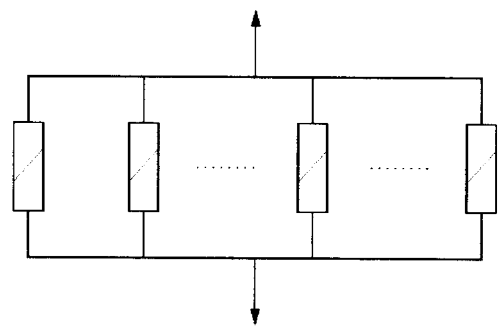


Figure 2. Parallel System

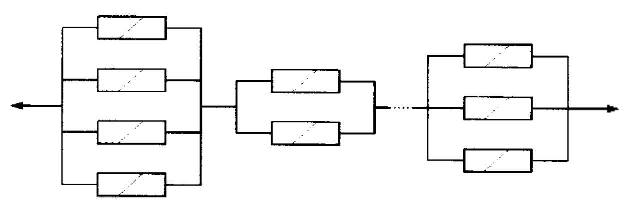


Figure 3. Hybrid System

(2) Target Requirements of the Methodology

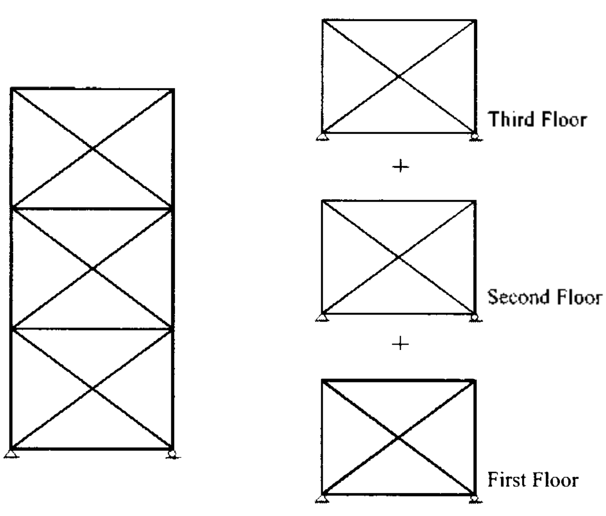
The target requirements for the proposed methodology are divided into two categories: primary requirements and functional requirements. Primary requirements are those characteristics and features that the methodology should possess. Functional requirements are specific tasks that must be performed by components of the methodology such that the primary requirements are satisfied. The desired primary requirements associated with the methodology can be stated as follows: (a) the reliability predictions should be accurate; (b) the rationale for the method should be logical; (c) the methodology should be internally consistent; (d) the methodology should be relevant; (e) the methodology should be versatile; (f) the methodology should not be overly complex; and (g) the methodology should be computationally competitive. These primary characteristics will be satisfied if the following functional requirements are performed:

1. The predicted probabilities of system failure should be close to values predicted by other theories;
2. The prediction should agree with Monte Carlo simulations of structural system failure;
3. How the proposed method is related to other methods of system reliability evaluations should be clearly indicated;
4. The method should be at least as computationally efficient as the other leading methods;
5. The methodology should apply to any structural type (e.g., frames, trusses, etc.…); and
6. The basic assumptions underlying the methodology should be clearly stated.

(3) Fundamental Assumptions Governing the Methodology

The proposed methodology builds on five fundamental assumptions that involve the following: (a) the definition of failure modes; (b) the extent of statistical correlation among safety margins of basic elements; (c) the sequence of component failure; (d) the definition of system failure; and (e) a rule for the rapid estimation of the failure probability of the system and the identification of the components of the most likely mode of failure. Theses basic assumptions are discussed below.

Types of Failure Modes: Two types of failure modes considered in this methodology can be discussed with the truss shown in Figure 4. The first type of failure mode is designated the "global failure mode". This type of mode exists when the structure as a whole becomes geometrically unstable. In Figure 4(a) for example, the degrees of static indeterminacy (redundancy) of the structure are three. Therefore, the structure will become kinematically unstable if any combination of four of its sixteen elements fail. In general, if s is the redundancy of a structure, the structure fails if s + 1 components fail.



(a) Whole System (b) Three Substructures of the System

Figure 4. A 16-Member Truss

Note that the structure may also be considered to have failed if anyone of the three stories fails. Figure 4(b) depicts three logical substructures of the complete system. Note that each substructure has a redundancy of one. Therefore, if any two elements of one of the substructures fail, a floor of the structure will fail. The latter failure modes are deemed local failure modes. Thus the number of failure modes for a structure is the sum of the global modes and the local modes. This discussion can be extended to other types of structures. All that one has to do is to determine the redundancy of the global structure, isolate the basic substructure units, then determine the redundancy of the substructures.

Justification of the Correlation Assumption: The basic unit in the reliability of a hybrid system is the failure mode. Note that each failure mode is a parallel system. The computation of the failure probability of a parallel system can be simplified significantly if it is assumed that the safety margins of the basic elements are completely correlated. This assumption holds true for many practical structures. Let a structure have M failure elements (e.g., the failure in tension of each and every one of the truss members of the truss shown in Figure 4). The random safety margin, g_i, of the i^{th} element is given by

$$g_i = R_i - \sum_{j=1}^{NL} a_{ij} S_j \tag{2}$$

where R_i is the random resistance of element i, S_j is the applied j^{th} load on the structure, a_{ij} is the influence coefficient which depends upon the geometry, redundancy, and material properties of the structure, and NL is the number of applied loads.

Let elements k, l, m, n, and o be involved in failure mode j. Then the probability of failure of that mode, P_j, is given by

$$P_j = P\{[g_k \le 0] \cap [g_l \le 0] \cap [g_m \le 0] \cap [g_n \le 0] \cap [g_o \le 0]\} \tag{3}$$

For most structures made of the same material, the resistance, R_i, are correlated. Also, because of the common existence of the loads S_j in Equation (2), the random safety margins are also correlated. Thus in most structures, one expects a significant degree of correlation among the safety margins. Therefore, if it is assumed that the safety margins of each and every mode are complete correlated, an estimate of the probability of failure of the mode given by Equation (3) is

$$P_j \approx \mathrm{Min}\{p_k, p_l, p_m, p_n, p_o\} \tag{4}$$

where, for example,

$$p_k = P[g_k \le 0] \tag{5}$$

Note that the estimate of the failure probability in Equation (4) represents an upper bound to the failure probability of a parallel system. This estimate is, therefore, a conservative estimate of the failure probability of the mode.

The Consequence of Ignoring the Sequence of Component Failure: Here it is asserted that the estimation of the failure probability of a highly redundant system can be significantly simplified if the sequence of the failure of the components of a given mode is ignored. If the latter assumption is adhered to, at least two major consequences follow. First, the number of failure modes to be considered is reduced significantly. For example, if there are a total of n failure components and each mode contains r elements, in the beta-unzipping method (Thoft-Christensen and Murotsu 1986) there are P(n,r) possible modes; whereas, if the sequence of the failure of the components is ignored, there are C(n,r) possible modes. Note that P(n,r) = r!C(n,r) where P denotes permutation and C denotes combination. For the truss example shown in Figure 4, for the case of global modes, r = 4 and n = 16. Accordingly, P(n,r) = 43,680 and C(n,r) = 1820.

The second consequence of the sequence-relaxation assumption is the reduction in the number of structural analyses that have to be performed to make a determination of the probability of failure of the system. In the beta-unzipping method, for example, if one wants to determine the most likely mode of failure for the truss in Figure 4, for s = 3, three structural analyses and re-analyses are required to determine the most "probable" sequence of failure. If the sequence of failure is ignored, only a single structural analysis is required.

The Probability of System Failure: Having introduced the simplifying assumptions of complete correlation among modes and the unimportance of the sequence of component failure in a given mode, the safety of a structure with many modes must now be defined. As noted previously, any hybrid system may be modeled as a series system of parallel modes such as the one depicted in Figure 3. Note that the failure probability of mode j is assigned to the structure according to the rule stipulated in Equation (4). If F_i is denoted to be the event that failure of mode i occurs and P_i is the probability of occurrence of that event, then the failure probability of the system is given by

$$P_{system} = P\left[\bigcup_{i=1}^{NM} F_i\right] \tag{6}$$

Since complete correlation among the modes has been assumed, it follows that

$$P_{system} = \mathrm{Max}\left[P_1, P_2, P_3, \ldots, P_{NM}\right] \tag{7}$$

Rapid Estimation of System Reliability: In order to execute Equation (7), one needs to know the probability of failure of each mode. Even though the sequence of component failure has been ignored, the task of identifying all NM possible failure modes could be formidable. For example, in the structure provided in Figure 4, the number of failure modes in the structure with a redundancy of as little as 3 and 16 possible failure elements is 1820. Thus the objective of this section is to develop a simple rule that engenders the solution of Equation (7) without having to identify the numerous individual failure modes. This objective can be accomplished via mathematical induction.

Let the system have n failure components. For example, in the truss depicted in Figure 4, the number of components is 16. Assuming that the component failure probabilities, a_i, are known, reorder the probabilities in descending order of magnitude such that $a_1 > a_2 > a_3 > \cdots > a_n$. Note that the magnitude of the probability a_1 is not necessarily the probability assigned to component 1. With this reordering, the author can make the following inferences:

1. If the failure model of the structure is a series system,

$$P_{system} = \mathrm{Max}\left[P_1, P_2, P_3, \ldots, P_{NM}\right] = a_1 \tag{8}$$

2. If the redundancy of the structure is one, the mode of maximum probability of failure contains components with failure probabilities a_1 and a_2. Consequently,

$$P_{system} = a_2 \tag{9}$$

3. If the redundancy of the structure is two, the mode with the maximum probability of failure contains the components with failure probabilities a_1, a_2, and a_3. Consequently,

$$P_{system} = a_3 \tag{10}$$

4. By mathematical induction, if the redundancy of the structure is s, the mode with the maximum probability of failure contains the components with failure probabilities $a_1, a_2, a_3, \ldots, a_s, a_{s+1}$. Consequently,

$$P_{system} = a_{s+1} \tag{11}$$

On the basis of Statement 4 above, the author propose the following rule to estimate the probability of failure of an arbitrary structure without a knowledge of the sequence of failure of the components in a mode or the elements of the mode.

If the failure probabilities of the elements are rearranged in order of descending magnitude $a_1 > a_2 > a_3 > \ldots\ldots > a_N$ and if the redundancy of the structure (or substructure) is s, an estimate of the failure probability of the system is given by $P_{system} = a_{s+1}$.

Furthermore, the s + 1 elements participating in the failure mode are those elements with failure probabilities $a_1, a_2, a_3, \ldots, a_s, a_{s+1}$. Note that the estimate given by Equation (11) may be thought of as a lower bound of the upper bound of failure mode probabilities.

(4) Estimation of Failure Probabilities of Components

Suppose a structural element has only one load effect S resisted by one resistance R. The load effect S may be obtained from the applied loading Q through a structural analysis. It is important to note that R and S should be expressed in the same units. If its resistance R is less than, or equal to, the stress resultant S acting on it, that element is considered to have failed. The probability of failure, p_f, of the structural element can be stated in any of the following ways

$$p_f = P(R \le S) = P(R - S \le 0) = P(R/S \le 1) \qquad (12)$$

or

$$p_f = P[G(R, S) \le 0] \qquad (13)$$

where G(•) is termed the "limit state function" or the "failure function".

The most notable example is when R and S are normal random variables with means μ_R and μ_S and variances σ_R^2 and σ_S^2, respectively. The safety margin M = R - S then has a mean and variance

$$\mu_M = \mu_R - \mu_S \qquad (14)$$

and

$$\sigma_M^2 = \sigma_R^2 + \sigma_S^2 \qquad (15)$$

using well-known rules for addition (subtraction) of normal random variables. Equation (12) then becomes

$$p_f = P(R - S \le 0) = P(M \le 0) = \Phi\left(\frac{0 - \mu_M}{\sigma_M}\right) \qquad (16)$$

where Φ is the standard normal distribution function (zero mean and unit variance). Using Equations (14) and (15) it follows that

$$p_f = \Phi\left(\frac{-(\mu_R - \mu_S)}{(\sigma_R^2 + \sigma_S^2)^{1/2}}\right) = \Phi(-\beta) \qquad (17)$$

where β is defined as the safety index (Cornell 1969)

$$\beta = \mu_M / \sigma_M \qquad (18)$$

Evidently, Equation (17) yields the exact probability of failure when both R and S are normally distributed and hence β is a direct measure of the safety of the structural element, i.e., greater β represents greater safety.

In Equation (17), only two random variables, i.e., load and resistance, were considered. If the limit state function G(**X**) consists of more than two basic random variables, a useful approach is to calculate approximate moments by expanding G($\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$) as a first-order Taylor series about the means $\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n}$ (Ang and Tang 1975)

$$\mu_M \approx G(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n}) \qquad (19)$$

and

$$\sigma_M^2 \approx \sum_{i=1}^{n}\left(\frac{\partial G}{\partial X_i}\right)^2\Big|_{\mu_{X_i}} \sigma_{X_i}^2 \qquad (20)$$

This is the well-known First-Order Second-Moment (FOSM) method. The safety index β defined by Equation (18) is still effective in this case and the probability of failure can also be formulated, provided the basic variables $\mathbf{X}_i$ = (i = 1, 2, ... , n) are normally distributed

$$p_f = \Phi(-\beta) \qquad (21)$$

In this study, Equations (19), (20), and (21) will be used as a mathematical model to estimate the failure probabilities of structural elements.

3. NUMERICAL EXAMPLES

The validity of the proposed methodology which can estimate the system reliability of arbitrary structure is demonstrated via numerical examples. It is assumed that the yield stresses of the members and the applied loads to the structure are statistically independent normal random variables. The objective of this section is to compare the exact reliability of a structure obtained using existing reliability theory with the reliability resulted from the proposed methodology. Three structures are selected for the purpose of this section. These structures include: (a) a redundant portal frame; (b) 6-member truss with one degree of redundancy; and (c) 16-member truss with three degrees of redundancy.

(1) Case I: Portal Frame

Reliability using the Existing Method: A simple portal frame is shown in Figure 5 (see Thoft-Christensen and Murotsu 1986). In this example, all joints are rigid moment-resisting joints. The height of the column is 5 meters and the span of the beam is 10 meters. Material properties are estimated such that all structural elements have the same area A = 0.459×10^{-2} m^2 and the same second moment of area I = 0.579×10^{-4} m^4. All members are assumed to be made of the same material with modulus of elasticity E = 0.21×10^9 kN/m^2, density ρ = 7850 kg/m^3, and Poisson's ratio ν = 0.3. The plastic bending moment capacities of the beam (R_b) and columns (R_c) are given such that

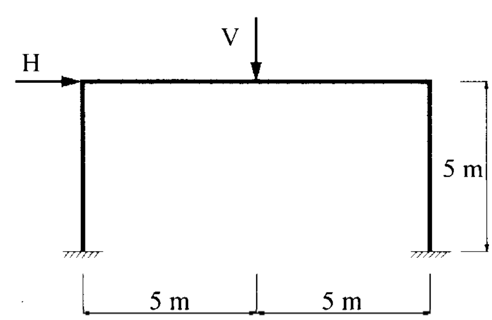


Figure 5. Geometry of Portal Frame

$$\mu_{R_b} = 135 \text{ kNm}, \quad \sigma_{R_b} = 13.5 \text{ kNm}$$
$$\mu_{R_c} = 135 \text{ kNm}, \quad \sigma_{R_c} = 13.5 \text{ kNm}$$

where μ denotes the mean and σ represents the standard deviation.

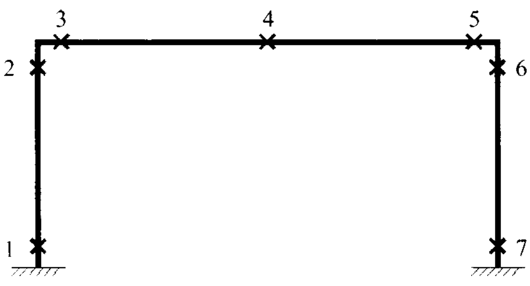


Figure 6. Potential Locations of Plastic Hinges

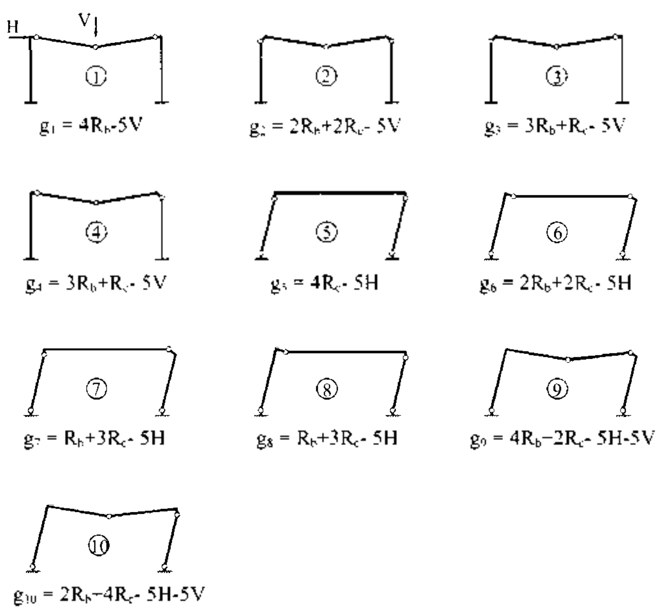


Figure 7. Possible Collapse Mechanisms

Table 1. Failure Probabilities for Possible Failure Modes

Mode No.	Failure Function	Safety Index (β)	Failure Probability (F_i)
1	$g_1 = 4R_b - 5V$	5.385	0.363×10^{-7}
2	$g_2 = 2R_b - 2R_c - 5V$	7.107	0.591×10^{-12}
3	$g_3 = 3R_b - R_c - 5V$	6.528	0.334×10^{-10}
4	$g_4 = 3R_b + R_c - 5V$	6.528	0.334×10^{-10}
5	$g_5 = 4R_c - 5H$	4.373	0.612×10^{-5}
6	$g_6 = 2R_b - 2R_c - 5H$	5.632	0.892×10^{-8}
7	$g_7 = R_b + 3R_c - 5H$	5.218	0.902×10^{-7}
8	$g_8 = R_b + 3R_c - 5H$	5.218	0.902×10^{-7}
9	$g_9 = 4R_b + 2R_c - 5H - 5V$	4.425	0.482×10^{-5}
10	$g_{10} = 2R_b + 4R_c - 5H - 5V$	4.425	0.482×10^{-5}

As shown in Figure 5, this portal frame is subjected to a random vertical load **V** and a random horizontal load **H**. The concentrated vertical load acts at the middle of the beam and the horizontal load is applied at the top of the left column. All loads are taken to be static. The vertical load **V** and horizontal load **H** are given by

$$\mu_V = 45 \text{ kN}, \quad \sigma_V = 4.5 \text{ kN}$$
$$\mu_H = 55 \text{ kN}, \quad \sigma_H = 5.5 \text{ kN}.$$

The collapse of the example structure may be caused by the formation of plastic hinges at the joints. The potential locations of plastic hinges are indicated by "×" in Figure 6. Ten possible collapse mechanisms resulting from the combination of the vertical load **V** and the horizontal load **H** are shown in Figure 7. The corresponding limit state functions (g_i) using the virtual work theory (McCormac 1975) are also presented in Figure 7. In each function, the work performed by the external loads during the displacement is subtracted from the internal work absorbed by the hinges. Failure is said to occur when this function is less than, or equal to, zero.

To compute the failure probabilities of the possible failure modes, the standard procedure from the First-Order Second-Moment (FOSM) reliability method summarized in the previous section is utilized. Table 1 lists the safety indices and the corresponding failure probabilities for the ten failure modes.

A simple bounds technique (Cornell 1967) is adopted to estimate the probability of failure of the system and then the lower and upper bounds are

$$\max(F_i) \le P_{system} \le 1 - \prod_{i=1}^{10}(1 - F_i) \approx \sum_{i=1}^{10} F_i \qquad (22)$$

where F_i is the probability of occurrence of mode i. Note that the lower bound assumes that all failure modes are fully dependent and upper bounds assumes that all failure modes are independent. From Table 1, the system failure probability is bounded by $0.612\times10^{-5} \le P_{system} \le 0.160\times10^{-4}$. Clearly, the failure modes 1, 2, 3, 4, 6, 7, and 8 have negligible effects on the failure probability of the system.

Reliability using the Proposed Method: The critical failure elements 1 to 7 are indicated by the symbol "×" in Figure 6. In this example, a joint fails when the resisting plastic moment capacity of the joint is exceeded. These critical elements are taken to be those locations where plastic hinges are most likely to occur in the structure. Note that the structure is statically indeterminate to the third degree. If the sequence of failure is ignored, failure of any four of the seven joints leads to the failure (kinematic instability) of the system. Therefore, there are thirty five possible combinations of seven element taken four at a time. The general failure model which describes this simple structure is therefore a system with thirty five, four-arm, parallel modes in series.

To compute the failure probabilities of the joints, the standard procedure from the First-Order Second-Moment (FOSM) reliability method is utilized. First, failure func-

Table 2. Component Failure Probabilities Using Proposed Methodology (Portal Frame)

Joint No.	Failure Function	Safety Index (β)	Failure Probability (P_i)
1	$g_1 = R_c - \lvert -1.5681H - 0.4919V \rvert$	4.384	0.584×10^{-5}
2	$g_2 = R_c - \lvert 0.9379H - 0.9973V \rvert$	8.478	0.115×10^{-16}
3	$g_3 = R_b - \lvert 0.9379H - 0.9973V \rvert$	8.478	0.115×10^{-16}
4	$g_4 = R_b - \lvert 0.0015H + 1.5027V \rvert$	4.457	0.415×10^{-5}
5	$g_5 = R_b - \lvert -0.9349H - 0.9973V \rvert$	2.558	0.526×10^{-2}
6	$g_6 = R_c - \lvert -0.9349H - 0.9973V \rvert$	2.558	0.526×10^{-2}
7	$g_7 = R_c - \lvert 1.5590H - 0.4919V \rvert$	1.680	0.465×10^{-1}

tions are generated for each joint. Beginning with the structure in Figure 5, one can first set **H** = 1 and **V** = 0. Next, a numerical structural analysis on the system can be performed and the resultant moment at each joint due to unit horizontal load can be set equal to a_{Hi}. Next, one can set **H** = 0 and **V** = 1, then perform a similar numerical structural analysis, and set the resultant moment at each joint due to unit vertical load to be b_{Vi}. With R_i as the resisting moment of joint *i*, the safety margin, g_i, for the joint becomes:

$$g_i = R_i - \left| a_{Hi} H + b_{Vi} V \right| \qquad (23)$$

where $\left| a_{Hi} H + b_{Vi} V \right|$ represents the absolute value of the resultant moment. These functions are provided in the second column of Table 2. Next, using the loading and resistance statistics with Equations (19)~(21), safety indices and the corresponding failure probabilities are listed in the third and fourth column of Table 2.

According to the rule proposed in this study, the system failure probability is the fourth largest value among the seven component failure probabilities. For example, the value of 0.584×10^{-5}, which is the fourth largest value in Table 2, is taken as the system failure probability.

(2) Case II: 6-Member Truss

Reliability using the Existing Method: Consider a statically indeterminate 6-member truss with one degree of redundancy shown in Figure 8 (Murotsu et al. 1980). The truss height is 91.44 cm and the width is 121.9 cm. Table 3 lists the material properties of truss members. As shown in Figure 8, one static concentrated force, P, is acting at the top of the left vertical member and the statistics of P are given such that $\mu_P = 44.45$ kN and $\sigma_P = 4.445$ kN.

Murotsu et al. (1980) defined that the complete failure of the structure is determined by investigating singularity of the total structure stiffness matrix K formed with the members in survival. That is, the criterion for complete failure is given by

$$\left| K \right| = 0 \qquad (24)$$

where $\left| K \right|$ represents the determinant of matrix K.

In case of a statically indeterminate truss, there exist many possible failure modes or paths to complete failure of the structure and it is impossible in practice to generate all of them. Hence, the failure probability is estimated by evaluating its lower and upper bounds. The lower bound is evaluated by selecting the dominant modes of failure and calculating their failure probabilities. The upper bound is evaluated by assuming that the redundant truss behaves itself like a statically determinate truss, i.e., the structure fails if any one member is subject to failure [See Murotsu et al. (1980) for detail]. Table 4 presents the lower bound and upper bound [Murotsu et al. (1980)] of the system failure probabilities along with the Monte Carlo simulation result performed by Kathir (1991) on the same structure. In this example, failure due to buckling is not considered.

Reliability using the Proposed Method: The failure element can be any member of 6 truss elements. In this example, a member fails when the resisting axial load capacity in tension or compression of the member is exceeded. Failure via buckling is not considered. Note that the structure is statically indeterminate to the one degree. If the sequence of failure is ignored, failure of any two of the six truss elements leads to the failure (kinematic instability) of the system. Therefore, there are fifteen possible combinations of six taken two at a time.

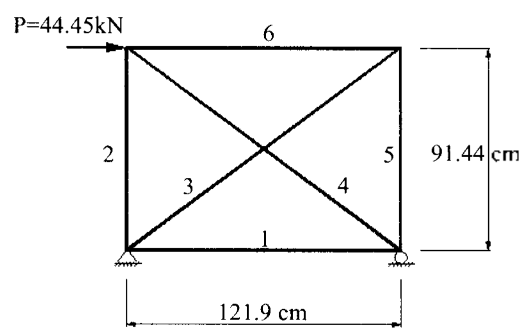


Figure 8. Statically Indeterminate 6-Member Truss with One Degree of Redundancy

Table 3. Material Properties for Truss Members (6-Member Truss)

Member Number	Cross-sectional area A (cm^2)	Mean value of yield stress, σ_y(Pa)	Elastic modulus, E (Pa)
1, 2, 5, 6	1.33	2.76×10^8	2.06×10^{11}
3, 4	1.49	2.76×10^8	2.06×10^{11}

Table 4. Failure Probabilities from Murotsu et al. (1980) and Kathir (1991)

COV		Failure Probability		
P	σ_y	Lower bound	Upper bound	Simulation
0.1	0.1	1.96×10^{-4}	6.40×10^{-3}	3.60×10^{-3}

To compute the failure probabilities of the failure elements, the standard procedure from the First-Order Second-Moment (FOSM) reliability method is utilized. First, failure functions are generated for each element. Beginning with the structure in Figure 8, set P = 1 and perform a numerical structural analysis on the system then set the resulting internal axial force at each member due to the unit load to be a_i. With V_i as the resisting axial capacity of element *i*, the safety margin, g_i, for the element becomes:

$$g_i = V_i - |a_i P| \qquad (25)$$

where $|a_i P|$ is the absolute value of the resultant axial force in tension or compression. These functions are provided in the second column of Table 5. Next, using the loading and resistance statistics with Equations (19)~(21), safety indices and the corresponding failure probabilities are listed in the third and fourth column of Table 5. According to the rule proposed in this study, the system failure probability is the second largest value among the six component failure probabilities. For example, the value of 3.59×10^{-3}, which is the second largest value in Table 5, is taken as the system failure probability.

(3) Case III: 16-Member Truss

Reliability using the Existing Method: Consider a statically indeterminate 16-member truss with three degree of redundancy shown in Figure 9 (See Murotsu et al. 1980). The truss width is 121.9 cm and consists of three stories in which the height of a story is 91.44 cm. Table 6 lists the material properties of truss members. As shown in Figure 9, three static concentrated forces denoted P are acting at the top of each story where μ_P = 44.45 kN and σ_P = 4.445 kN.

The failure modes are the same as described in the example of a 6-member truss. Table 7 presents the lower bound and upper bound [Murotsu et al. (1980)] of the system failure probabilities. In this example, failure due to buckling is not considered.

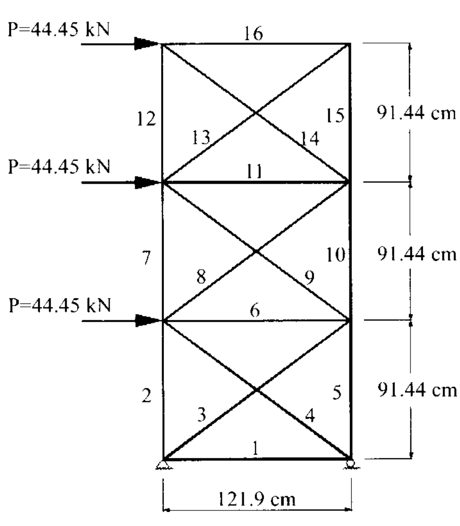


Figure 9. Statically Indeterminate 16-Member Truss with Three Degrees of Redundancy

Table 5. Component Failure Probabilities Using Proposed Methodology (6-Member Truss)

Element Number	Failure Function	Safety Index (β)	Failure Probability (P_f)
1	$g_1 = V_1 - \|0.50P\|$	3.375	3.69×10^{-4}
2	$g_2 = V_2 - \|0.3751P\|$	4.969	3.36×10^{-7}
3	$g_3 = V_3 - \|0.6250P\|$	2.689	3.59×10^{-3}
4	$g_4 = V_4 - \|-0.6250P\|$	2.689	3.59×10^{-3}
5	$g_5 = V_5 - \|-0.3751P\|$	4.969	3.36×10^{-7}
6	$g_6 = V_6 - \|-0.50P\|$	3.375	3.69×10^{-4}

Table 6. Material Properties for Truss Members (16-Member Truss)

Member Number	Cross-sectional area A (cm^2)	Mean value of yield stress, σ_y(Pa)	Elastic modulus, E (Pa)
1	3.35		
2, 5	8.64		
3, 4, 14	5.76		
6	2.29	2.76×10^8	2.06×10^{11}
7, 8, 10	4.03		
9	7.35		
11, 12, 15	1.58		
13, 16	2.29		

Table 7. Failure Probabilities from Murotsu et al. (1980)

COV		Failure Probability	
P	σ_y	Lower bound	Upper bound
0.1	0.1	2.73×10^{-6}	3.84×10^{-3}

Reliability using the Proposed Method: The failure element can be any member of 16 truss elements. In this example, a member fails when the resisting axial load capacity in tension or compression of the member is exceeded. Failure via buckling is not considered. The system may fail as a result of global instability or local instability. First, the global instability is examined. Note that the structure is statically indeterminate to the third degree. If the sequence of failure is ignored, failure of any four of the sixteen truss elements leads to the failure (kinematic instability) of the system. Therefore, there are 1820 possible combinations of sixteen taken four at a time. As illustrated in Figure 4, one may define local failure if anyone of the three stories fails. Therefore, if any two elements of one of the substructures fail, a floor of the structure will fail.

The failure functions for each element are generated using Equation (25). These functions are provided in the second column of Table 8. Next, using the loading and resistance statistics with Equations (19)~(21), safety indices and the corresponding failure probabilities are listed in the third and fourth column of Table 8. According to the rule proposed in this study, the system failure probability is the fourth largest value among the sixteen component failure probabilities for the global failure modes. For example, the value of 2.76×10^{-4}, which is the fourth largest value in Table 8, is taken to be the system failure probability. For the local modes, the failure probability for each floor is given by the second largest failure probability of the elements. For example, the failure probability for the first floor is 5.04×10^{-4}, the failure probability for the second floor is 2.37×10^{-6}, and the failure probability for the third floor is 5.14×10^{-13}. Therefore, if the local and global modes are considered together, the failure probability of the system is 5.04×10^{-4}.

Table 8. Component Failure Probabilities Using Proposed Methodology (16-Member Truss)

Element Number	Failure Function	Safety Index (β)	Failure Probability (P_i)
1	$g_1 = V_1 - \|1.2430P\|$	3.455	2.76×10^{-4}
2	$g_2 = V_2 - \|3.1828P\|$	3.498	2.34×10^{-4}
3	$g_3 = V_3 - \|2.1964P\|$	3.288	5.04×10^{-4}
4	$g_4 = V_4 - \|-1.5538P\|$	5.187	1.07×10^{-7}
5	$g_5 = V_5 - \|-3.5684P\|$	2.788	2.65×10^{-3}
6	$g_6 = V_6 - \|-0.3929P\|$	6.975	1.52×10^{-12}
7	$g_7 = V_7 - \|1.7734P\|$	2.377	8.73×10^{-3}
8	$g_8 = V_8 - \|0.7949P\|$	6.503	3.93×10^{-11}
9	$g_9 = V_9 - \|-1.7052P\|$	5.867	2.21×10^{-9}
10	$g_{10} = V_{10} - \|-1.2271P\|$	4.576	2.37×10^{-6}
11	$g_{11} = V_{11} - \|0.0648P\|$	9.319	5.86×10^{-21}
12	$g_{12} = V_{12} - \|0.5257P\|$	4.091	2.15×10^{-5}
13	$g_{13} = V_{13} - \|0.3741P\|$	7.127	5.14×10^{-13}
14	$g_{14} = V_{14} - \|-0.8760P\|$	7.334	1.12×10^{-13}
15	$g_{15} = V_{15} - \|-0.2245P\|$	7.517	2.79×10^{-14}
16	$g_{16} = V_{16} - \|-0.2992P\|$	7.727	5.52×10^{-15}

4. DISCUSSION OF RESULTS

In Table 9, the estimated failure probabilities generated using the proposed methodology are compared to those calculated from existing reliability techniques. Note that the basic assumptions behind the methodology were: (a) the safety margins of the elements and failure modes are fully correlated; (b) the kinematic instability of a structural system defines the failure or collapse of the structure; and (c) the order of the failure of components in a given mode is disregarded. Certainly, investigations and comparisons on other structures to confirm the assumptions must be produced in the future. From the results, the following observations are made:

1. The predicted probabilities of system failure using the proposed technique close to or fall within bounds proposed in the literature;
2. In the case where data were available, the predicted failure probability agrees well with the result obtained via Monte Carlo simulation;
3. The proposed technique is computationally efficient when compared to other methods, especially for the complex structures that are highly indeterminate and have many elements;
4. The proposed technique automatically seeks and identifies the most probable failure mode; and
5. The proposed technique can be applied to any structural type.

5. SUMMARY AND CONCLUSIONS

In this paper, a general methodology for the estimation of systems reliability for arbitrary structural systems was proposed. First, a theory to estimate the system failure probability was developed under the basic assumptions that the failure modes are completely correlated and that the sequence of component failure in a given mode is not important. According to the consequence of these assumptions, a simple rule has been proposed for direct estimation of system failure probability from the element or component failure probabilities. A model to estimate the element or component failure probabilities was formulated by the First-Order Second-Moment (FOSM) reliability method. Finally, the proposed technique was compared to the current reliability methods (i.e., bound technique and Monte Carlo simulation) through three examples: (1) a simple portal frame; (2) a six-member truss; and (3) a sixteen-member truss.

From this study, two conclusions are made. First, the results obtained for the example structures used here indicate that the predicted system reliability values agree closely with those values predicted by other approaches. Second, the proposed technique is computationally efficient when compared to other methods, especially for the complex structures that are highly indeterminate and have many elements.

Table 9 Comparisons of Reliability Techniques

Type of Structure	Probability of Failure of System			
	Bounds Technique		Monte Carlo	Proposed
	Lower	Upper	Simulation	Technique
Portal Frame	6.12×10^{-6}	1.60×10^{-5}	-	5.84×10^{-6}
6-Member Truss	1.96×10^{-4}	6.40×10^{-3}	3.60×10^{-3}	3.59×10^{-3}
16-Member Truss	2.73×10^{-4}	3.84×10^{-3}	-	5.04×10^{-4}

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