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R–MAPS AND L–MAPS IN BH–ALGEBRAS

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Abstract. In this paper, we introduce the notion of positive implicative BH-algebras and study some relations between $R-(L-)$ maps and positive implicativity in BH-algebras.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([2, 3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. BCK-algebras have some connections with other areas: D. Mundici ([8]) proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras, and J. Meng ([6]) proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. In [5], Y. B. Jun, E. H. Roh and H. S. Kim introduced the new notion, called an BH-algebra, which is a generalization of $BCH/BCI/BCK$-algebras. They defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. Furthermore, they constructed the quotient BH-algebras via translation ideals and obtained the fundamental theorem of homomorphisms

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for BH-algebras as a consequence. In this paper, we introduce the notion of positive implicative BH-algebras and study some relations between $R-(L-)$ maps and positive implicativity in BH-algebras.

2. BH-algebras

In 1983, Q. P. Hu and X. Li ([1]) introduced a very interesting class of algebras, called a BCH-algebra. An algebra $(X;*,0)$ of type (2,0) with the following axioms: for all $x,y,z\in X$,

(1) $x*x=0$,

(2) $(x*y)*z=(x*z)*y$,

(3) $x*y=0$ and $y*x=0$ imply $x=y$,

is called a *BCH-algebra*. It is well known that for any BCH-algebra X

(4) $x*0=x$ for all $x\in X$.

DEFINITION 2.1. ([5]) By a *BH-algebra*, we mean an algebra $(X;*,0)$ of type (2,0) satisfying the conditions (1), (3) and (4).

EXAMPLE 2.2. ([5]) (a) Let $X:=\{0,1,2,3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

Then $(X;*,0)$ is a BH-algebra.

(b) Let $\mathbb{R}$ be the set of all real numbers and define

$$x*y:=\begin{cases}0 & \text{if } x=0,\\ \frac{(x-y)^2}{x} & \text{otherwise,}\end{cases}$$

for all $x, y \in \mathbb{R}$, where "$-$" is the usual substraction of real numbers. Then $(\mathbb{R}; *, 0)$ is a BH-algebra.

Let X and Y be BH-algebras. A mapping $f : X \to Y$ is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for any $x, y \in X$. A homomorphism f is called a *monomorphism* (resp., *epimorphism*) if it is injective (resp., surjective). A bijective homomorphism is called an *isomorphism*. Two BH-algebras X and Y are said to be *isomorphic*, written $X \cong Y$, if there exists an isomorphism $f : X \to Y$. For any homomorphism $f : X \to Y$, the set $\{x \in X | f(x) = 0\}$ is called the *kernel* of f, denoted by $Ker(f)$, and the set $\{f(x)|x \in X\}$ is called the *image* of f, denoted by $Im(f)$. Notice that $f(0) = 0$ for any homomorphism f.

3. R-maps and L-maps in BH-algebras

In this section, we define R-maps and L-maps in BH-algebras and investigate several properties in BH-algebras.

DEFINITION 3.1. Let X be a BH-algebra. For a fixed $a \in X$, we define a map $R_a : X \to X$ such that $R_a(x) := x * a$ for all $x \in X$, and call R_a a *right map* on X. The set of all right maps on X is denoted by $\mathbb{R}(X)$. A *left map* is defined by a similar way, and the set of all left maps on X is denoted by $\mathbb{L}(X)$.

DEFINITION 3.2. A right map R_a is said to be *idempotent* if $R_a \circ R_a = R_a$, i.e., $(x * a) * a = x * a$ for all $x \in X$.

DEFINITION 3.3. A BH-algebra $(X; *, 0)$ is said to be *positive implicative* if it satisfies for all x, y and $z \in X$,

$$(x * z) * (y * z) = (x * y) * z.$$

EXAMPLE 3.4. Let $X := \{0, a, b, 1\}$ be a set with the following table:

*	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	1	1	0

Then $(X; *, 0)$ is a positive implicative BH-algebra.

LEMMA 3.5. *Let $X := (X; *, 0)$ be a BH-algebra. If X is positive implicative, then the following condition holds:*

$$(x * y) = (x * y) * y \quad \textit{for any} \quad x, y \in X.$$

Proof. For any $x, y \in X$, $x*y = (x*y)*0 = (x*y)*(y*y) = (x*y)*y$, since X is positive implicative. This completes the proof. □

THEOREM 3.6. *If a BH-algebra X is positive implicative, then every right map on X is idempotent.*

Proof. Since X is positive implicative, $x * y = (x * y) * y$ for any $x, y \in X$, by Lemma 3.5. Hence $R_y(x) = (R_y \circ R_y)(x)$ and so $R_y = R_y \circ R_y$ for any $y \in X$. □

The converse of Theorem 3.6 need not be true in BH-algrbras.

EXAMPLE 3.7. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	1	0

Then $(X;*,0)$ is a BH-algebra which is not a BCK-algebra. Since X satisfies $(x*y)*y=(x*y)$ for any $x,y\in X$, every right map on X is idempotent. But X is not positive implicative, since $(3*1)*2=3\neq 0=1*1=(3*2)*(1*2)$.

THEOREM 3.8. *A BH-algebra X is positive implicative if and only if every right map on X is an endomorphism of X.*

Proof. If X is a positive implicative BH-algebra, then for each $a\in X$, $(x*y)*a=(x*a)*(y*a)$, i.e., $R_a(x*y)=R_a(x)*R_a(y)$. Hence R_a is an endomorphism. The converse follows immediately. The proof is complete. □

PROPOSITION 3.9. *Let $f:X\to Y$ be a homomorphism of BH-algebras. Then f is injective if and only if $Ker f=\{0\}$.*

Proof. Straightforward. □

THEOREM 3.10. *If L_x is a homomorphism, $x\in X$, then $x=0$.*

Proof. For any $x\in X$, we have

$$x=x*0=L_x(0)=L_x(0*0)=L_x(0)*L_x(0)=0.$$

□

The converse of Theorem 3.10 need not be true in BH-algebras in general. In Example 2.2, $L_0(1*3)=0*(1*3)=0*0=0$ and $L_0(1)*L_0(3)=(0*1)*(0*3)=3*2=1$. Hence $L_0(1*3)\neq L_0(1)*L_0(3)$. Thus L_0 is not a homomorphism.

For a positive implicative BH-algebra X, we define an operation $\circledast$ in $\mathbb{L}(X)$ as follows. For any $L_a,L_b\in\mathbb{L}(X)$ and any $x\in X$,

$$(L_a\circledast L_b)(x):=L_a(x)*L_b(x).$$

Using positive implicativity of X, we know

$$(L_a \circledast L_b)(x) = (a * x) * (b * x) = (a * b) * x = L_{a*b}(x),$$

so $L_a \circledast L_b \in \mathbb{L}(X)$.

The next theorem gives a characterization of a positive implicative BH-algebra by its left maps.

THEOREM 3.11. *If X is a positive implicative BH-algebra, then $\mathbb{L}(X)$ is a positive implicative BH-algebra and X is isomorphic to $\mathbb{L}(X)$.*

Proof. For any $x \in X$, by positive implicativity of X we have

$$\begin{aligned}((L_a \circledast L_b) \circledast L_c)(x) =& ((a * x) * (b * x)) * (c * x)\\ =& ((a * x) * (c * x)) * ((b * x) * (c * x))\\ =& ((L_a \circledast L_c)(x)) * ((L_b \circledast L_c)(x))\\ =& ((L_a \circledast L_c) \circledast (L_b \circledast L_c))(x)\end{aligned}$$

which means

$$(L_a \circledast L_b) \circledast L_c = (L_a \circledast L_c) \circledast (L_b \circledast L_c).$$

It is easy to check that $(\mathbb{L}(X); \circledast, L_0)$ is a BH-algebra. Therfore it is a positive implicative BH-algebra. Next, we show that a map $f : X \to \mathbb{L}(X)$ defined by $f(x) := L_x$ is an isomorphism. Suppose that $f(x) = f(y)$, i.e., $L_x = L_y$ and so for any $t \in X$, $L_x(t) = L_y(t)$ and hence $x * t = y * t$. If we put $t = y$, then $x * y = y * y = 0$. Similarly, $y * x = 0$. Since X is a BH-algebra, $x = y$. This means that f is injective. Clearly f is also surjective. Since for any $t \in X$, $f(x * y)(t) = L_{x*y}(t) = (x * y) * t = (x * t) * (y * t) = L_x(t) * L_y(t) = (L_x \circledast L_y)(t) = (f(x) \circledast f(y))(t)$. It follows that f is a homomorphism. This proves the theorem. □

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