JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 13, No.2, December 2000

A CHARACTERIZATION OF THE N–DIMENSIONAL UNIT SPHERE IN THE 2N–DIMENSIONAL DE SITTER SPACE

Joonsang Park*

Abstract. We show that an n-dimensional unit sphere in the $2n$-dimensional de Sitter space can be associated to a solution of a partial differential equation on a Lorentzian Grassmannian system coming from the integrable system theory.

1. Introduction

The problem of immersions of space forms into space forms is one of the important and interesting question in the classical differential geometry. A well-known theorem of Hilbert states that a complete 2-dimensional Riemannian manifold of constant negative curvature, say, the hyperbolic space form $\mathbb{H}^2$ cannot be isometrically immersed into 3-dimensional Euclidean space $\mathbb{R}^3$ [3]. The natural generalization of the Hilbert theorem that a complete n-dimensional hyperbolic space form $\mathbb{H}^n$ cannot be isometrically immersed into $\mathbb{R}^{2n-1}$ is not known until now. The local problem of isometric immersions of space forms in space forms was studied by Cartan [2]. He showed that there is no immersion of $\mathbb{H}^n$ in $\mathbb{R}^{2n-2}$, and he constructed an example of a local immersion of $\mathbb{H}^n$ in $\mathbb{R}^{2n-1}$.

---

*This work was supported by Grant No. 2000-2-10100-002-3 from the Basic Research Program of the Korea Science & Engineering Foundation.

Received by the editors on November 1, 2000.

1991 *Mathematics Subject Classifications*: Primary 57N35.

Key words and phrases: isometric immersions, Grassmannian systems, flat connections.

Recently, Terng [7] studied local immersions of space forms in space forms with the same constant curvatures, or,

$$\mathbb{R}^n \text{ in } \mathbb{R}^{2n},\ \mathbb{S}^n \text{ in } \mathbb{S}^{2n},\ \mathbb{H}^n \text{ in } \mathbb{H}^{2n}.$$

She introduced a partial differential equation called an n-dimensional system on a Lie group or on a symmetric space and showed that by properly choosing symmetric spaces, the solutions of n-dimensional systems on those symmetric spaces correspond to such local immersions. We will call these n-dimensional systems as Grassmannian systems. This work was extensively generalized by Brück, Du, Park and Terng [1] later. The basic idea of these works [7] and [1] is that the fundamental equations of such immersions can be expressed by flat connections on the ambient spaces N with values in the Lie algebras $\mathcal{G}$ of the isometry groups of N, and these connections correspond to Lax pairs on $\mathcal{G}$. Thus it is plausible to expect that these phenomena of integrable systems can be applied to submanifolds in Lorentzian space.

The main goal of this paper is to show that locally the nondegenerate immersions of Riemannian manifolds S^n of constant curvature 1 with flat normal bundles in the $2n$-dimensional de Sitter space $S^{2n-1,1}$ correspond to the solutions of the system on $O(2n,1)/(O(n+1) \times O(n-1,1))$.

2. Submanifolds in Lorentzian space

First, we introduce basic knowledge about Lorentzian geometry and notations. For details, see [5] and [6]. Lorentzian space $\mathbb{R}^{n,1}$ is the vector space $\mathbb{R}^{n+1}$ with the nondegenerate metric $< x, y >= x_1y_1 \cdots + x_ny_n - x_{n+1}y_{n+1}$.

Suppose $X : M^n \to \mathbb{R}^{n+k,1}$ is a Riemannian isometric immersion.

A local orthonormal frame field $e_1, \cdots, e_{n+k+1}$ in $\mathbb{R}^{n+k,1}$ is said to be *adapted* to M, if when restricted to M, $e_1, \cdots, e_n$ are tangent to

M. From now on, we shall use the following index convention:

$$1 \le A, B, C \le n+k+1, \quad 1 \le i, j, k \le n, \quad n+1 \le \alpha, \beta, \gamma \le n+k+1.$$

Let ω_A be the dual coframe on $\mathbb{R}^{n+k,1}$, that is, $\omega_A(e_B) = \epsilon_A \delta_{AB}$, where, $\epsilon_A =< e_A, e_A >$. Thus the first fundamental form on M is given by $I = \sum_i \omega_i \otimes \omega_i$. Let ω_{AB} be the connection 1-form corresponding to the usual differential d on $\mathbb{R}^{n+k,1}$,

$$de_A = \sum_B \omega_{AB} \otimes e_B.$$

This induces the Levi-Civita connection ∇ on M by

$$\nabla e_i = \sum_j \omega_{ij} \otimes e_j, \qquad \nabla \omega_i = \sum_j \omega_{ij} \otimes \omega_j,$$

and the structure equations on M are

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j. \tag{2.1}$$

The Gauss, Codazzi and Ricci equations are

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \sum_\alpha \omega_{i\alpha} \wedge \omega_{\alpha j} - c\, \omega_i \wedge \omega_j, \tag{2.2}$$

$$d\omega_{i\alpha} = \sum_k \omega_{ik} \wedge \omega_{k\alpha} + \sum_\beta \omega_{i\beta} \wedge \omega_{\beta\alpha}, \tag{2.3}$$

$$d\omega_{\alpha\beta} = \sum_i \omega_{\alpha i} \wedge \omega_{i\beta} + \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}. \tag{2.4}$$

¿From (2.3) and (2.4), we get the curvature 2-form Ω on M and the normal curvature 2-form Ω^ν as

$$\Omega_{ij} = \sum_\alpha \omega_{i\alpha} \wedge \omega_{j\alpha} + c\, \omega_i \wedge \omega_j, \tag{2.5}$$

$$\Omega^{\nu}_{\alpha\beta} = \sum_i \omega_{i\alpha} \wedge \omega_{i\beta}. \tag{2.6}$$

It is an elementary theorem that M has the constant sectional curvature c if and only if

$$\Omega_{ij} = c\, \omega_i \wedge \omega_j. \tag{2.7}$$

The shape operator A_{e_α} in the direction e_α is defined by

$$A = \sum_{j,\alpha} \epsilon_\alpha \omega_{j\alpha} \otimes \omega_\alpha \otimes e_j, \tag{2.8}$$

which is identified with the second fundamental form II under the metric isomorphism $T^*\mathbb{R}^{n+k,1} \simeq T\mathbb{R}^{n+k,1}$:

$$\mathit{II} = \sum_{j,\alpha} \omega_{j\alpha} \otimes \omega_j \otimes e_\alpha. \tag{2.9}$$

Now, suppose the normal bundle is flat, i.e., $\Omega^\nu = 0$. Then there exists a parallel normal frame e_α and it is easy to see that all the shape operators commute by (2.6), and thus they are simultaneously diagonalizable.

Definition. A submanifold M^n is called *nondegenerate* if

$$\dim\{A_v | v \in \nu_x M\} = n.$$

Proposition 2.1. *Suppose M^n of $\mathbb{R}^{n+k,1}$ is a nondegenerate submanifold with the flat normal bundle. Then $n \leq k+1$.*

Proof. Since $\Omega^\nu = 0$, we can take an orthonormal basis e_i of which are common eigenvectors of A_v for any $v \in \nu_x M$. According to this basis, A_v is identified with a set of diagonal matrices. Since M^n is nondegenerate, the common eigenvalues of A are linearly independent functions on $\nu_x M$ and thus $k+1 = \dim \nu_x M \geq \dim\{A_v\}$. $\square$

Hence $k+1$ is the minimal codimension for nondegenerate isometric immersions with the flat normal bundle.

DEFINITION. The de Sitter space $\mathbb{S}^{n+k-1,1}$ in $\mathbb{R}^{n+k,1}$ is defined by

$$\mathbb{S}^{n+k-1,1} = \{x \in \mathbb{R}^{n+k,1} | < x, x >= 1\}.$$

It is well-known that $\mathbb{S}^{n+k-1,1}$ is a pseudo-Riemannian manifold with constant sectional curvature 1.

Now, we describe the geometry of the unit sphere $\mathbb{S}^n$ in $\mathbb{S}^{2n-1,1}$ which is nondegenerate and has the flat normal bundle.

THEOREM 2.2. *Let $X : \mathcal{O} \subset \mathbb{S}^n \longrightarrow \mathbb{S}^{2n-1,1}$ be a nondegenerate local immersion of the Riemannian manifold $\mathbb{S}^n$ of constant sectional curvature 1 with the flat normal bundle. Then there exist a coordinate system $x = (x_1, \cdots, x_n)$ on $\mathcal{O}$, a normal frame e_α, $b = (b_1, \cdots, b_n) : \mathcal{O} \longrightarrow \mathbb{R}^n$ and an $n \times n$ matrix $A = (a_{ij}) : \mathcal{O} \longrightarrow SO(n-1,1)$ such that the first and second fundamental forms are given by*

$$I = \sum_i b_i^2 dx_i^2, \tag{2.10}$$

$$I\!I = \sum_{i,j} a_{ij}^{-1} b_i dx_i^2 \otimes e_{n+j}. \tag{2.11}$$

Proof. Since $\Omega^\nu = 0$, we can choose a parallel normal frame e_α and the common principal directions e_i. The result follows from the nondegeneracy and the argument as in [2] or [4]. □

3. Lorentzian Grassmannian systems

G/K systems are introduced in [7] as first flows of integrable systems of evolution, where all the variables are on equal footing to play roles as time variables and the first flows in each variable is a first flow. We will review definitions briefly. For more details, see [7].

Let G/K be a rank n symmetric space, $\sigma : \mathcal{G} \to \mathcal{G}$ the corresponding involution, $\mathcal{G} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition, and $\mathcal{A} \subset \mathcal{P}$ a maximal abelian subalgbra. Let $a_1, \ldots, a_n$ be a basis for $\mathcal{A}$ consisting of regular elements with respect to the $Ad(K)$-action on $\mathcal{P}$. Let $\mathcal{A}^\perp$ denote the orthogonal complement of $\mathcal{A}$ in $\mathcal{G}$ with respect to the Killing form. Then G/K *system* for $v : R^n \to \mathcal{P} \cap \mathcal{A}^\perp$ is

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] = \big[\, [a_i, v], [a_j, v] \,\big], \qquad 1 \le i \ne j \le n,$$

where, $v_{x_i} = \frac{\partial v}{\partial x_i}$. This system is equivalent to the following Lax pair:

$$\Big[\frac{\partial}{\partial x_i} + \lambda a_i + [a_i, v], \frac{\partial}{\partial x_j} + \lambda a_j + [a_j, v]\Big] = 0, \quad \forall \lambda \in C.$$

The Cauchy problem for G/K system can be solved for any generic data decaying rapidly along $(x_1, 0, \ldots, 0)$ (cf. [7]).

We can also express G/K system in terms of a connection 1-form on the trivial principal bundle $R^n \times \mathcal{G}$ on R^n. To see this, we need the following proposition, which can be proved by a direct computation. We assume all Lie groups are subgroups of $GL(n)$ in this paper.

Proposition 3.1. *Given smooth maps $A_i : R^n \to \mathcal{G}$ for 1$\le i \le n$, the following statements are equivalent:*

(1) $E_{x_i} = EA_i$ *is solvable for* $E : R^n \to G$,

(2) $[\frac{\partial}{\partial x_i} + A_i, \frac{\partial}{\partial x_j} + A_j] = 0$,

(3) $(A_j)_{x_i} - (A_i)_{x_j} + [A_i, A_j] = 0$,

(4) $d\theta + \theta \wedge \theta = 0$, *where θ is the $\mathcal{G}$-valued 1-form* $\sum_{i=1}^n A_{x_i} dx_i$.

In this case, we call E a trivialization of θ and it satisfies $E^{-1}dE = \theta$.

Suppose E is a trivialization of θ, $E^{-1}dE = \theta$. Let $g : R^n \to G$. The gauge transformation of E by g is defined as $g * E = Eg^{-1}$. This induces a new flat connection

$$(Eg^{-1})^{-1} d(Eg^{-1}) = g\theta g^{-1} - dgg^{-1}.$$

We call $g * \theta = g\theta g^{-1} - dgg^{-1}$, the gauge transformation of θ by g.

It is easy to see that v is a solution for G/K system if and only if the following one-parameter family of $\mathcal{G}_C$-valued connections on R^n is flat:

$$\Theta_\lambda = \sum_{i=1}^{n} (a_i \lambda + [a_i, v]) dx_i.$$

Now, we introduce the system on $O(2n,1)/(O(n+1) \times O(n-1,1))$.

Let $\mathcal{G} = so(2n,1)$ and $\sigma : \mathcal{G} \to \mathcal{G}$ be an involution defined by $\sigma(X) = I^{-1}_{n+1,n} X I_{n+1,n}$, where $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$.

Then the Cartan decomposition is $\mathcal{G} = \mathcal{K} + \mathcal{P}$, where

$$\mathcal{K} = \left\{ \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \middle| \, Y_1 \in so(n+1), Y_2 \in so(n-1,1) \right\},$$

$$\mathcal{P} = \left\{ \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & b \\ -JF^t & -Jb^t & 0 \end{pmatrix} \middle| \, F \in gl(n), \; b \in \mathcal{M}_{n\times 1} \right\}.$$

Here, we denote by $\mathcal{M}_{p\times q}$ the set of $p \times q$ matrices and $J = \mathrm{diag}(\epsilon_1, \cdots, \epsilon_n)$, $-\epsilon_n = \epsilon_1 = \cdots = \epsilon_{n-1} = 1$. It is easy to see that

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & 0 & CJ \\ 0 & 0 & 0 \\ -C & 0 & 0 \end{pmatrix} \middle| \, C \in gl(n) \text{ is diagonal} \right\}$$

is a maximal abelian subalgebra in $\mathcal{P}$. Put

$$a_i = \begin{pmatrix} 0 & 0 & C_i J \\ 0 & 0 & 0 \\ -C_i & 0 & 0 \end{pmatrix},$$

where, C_i is the diagonal matrix whose i-th entry is 1 and 0 elsewhere. Then $a_1, \ldots, a_n$ form a basis of $\mathcal{A}$.

Let $gl_*(n) = \{(x_{ij}) \in gl(n) \mid x_{ii} = 0,\ 1 \leq i \leq n\}$. Then any element $v \in \mathcal{P} \cap \mathcal{A}^\perp$ is expressed as

$$v = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & b \\ -JF^t & -Jb^t & 0 \end{pmatrix},$$

where, $F \in gl_*(n)$.

By a direct calculation, we obtain

PROPOSITION 3.2. *$v : R^n \to \mathcal{P} \cap \mathcal{A}^\perp$ is a solution of the system $O(2n,1)/(O(n+1) \times O(n-1,1))$ if and only if v satisfies*

$$\begin{cases} (f_{ij})_{x_i} + (f_{ji})_{x_j} = \sum_{k=1}^n f_{ki} f_{kj} + b_i b_j, & \text{for } i \neq j, \\ (f_{ij})_{x_k} + f_{ik} f_{kj} = 0, & \text{for distinct } i, j, k, \\ (b_j)_{x_i} + b_i f_{ij} = 0, & \text{for } i \neq j, \\ \epsilon_j (f_{ij})_{x_j} + \epsilon_i (f_{ji})_{x_i} = \sum_{k=1}^n \epsilon_k f_{ik} f_{jk}, & \text{for } i \neq j. \end{cases}$$

The corresponding 1-parameter family of flat connection 1-forms is

$$\begin{aligned} \Theta_\lambda &= \sum_{i=1}^n (a_i \lambda + [a_i, v]) dx_i \\ &= \sum_i \begin{pmatrix} FC_i - C_i F^t & -C_i b^t & \lambda C_i J \\ bC_i & 0 & 0 \\ -\lambda C_i & 0 & JF^t C_i J - C_i F \end{pmatrix} dx_i \\ &= \begin{pmatrix} -\omega & -\delta b^t & \lambda \delta J \\ b\delta & 0 & 0 \\ -\lambda\delta & 0 & \eta \end{pmatrix}, \end{aligned} \tag{3.1}$$

where, $\delta = \operatorname{diag}(dx_1, \cdots, dx_n)$, $\omega = -(\delta F - F^t \delta)$ and $\eta = J\delta F^t J - F\delta$.

In this case, since v is determined by (F, b), we will say that (F, b) is a solution of this system instead of v being a solution. We restate Proposition 3.2 in terms of (F, b);

THEOREM 3.3. *(F,b) is a solution of $O(2n,1)/(O(n+1)\times O(n-1,1))$ system if and only if (F,b) satisfies*

$$
\left\{
\begin{aligned}
& d\eta+\eta\wedge\eta=0,\\
& d\omega=\omega\wedge\omega-\delta b\wedge b^t\delta,\\
& \delta\wedge db+\omega\wedge\delta b=0,\\
& \delta\wedge\eta=\omega\wedge\delta.
\end{aligned}
\right.
\tag{3.2}
$$

4. Main Theorems

Let X be a local immersion of $\mathbb{S}^n$ in $\mathbb{S}^{2n-1,1}$ as in theorem 2.2. Choose a tangent frame $e_i=\frac{1}{b_i}\frac{\partial}{\partial x_i}$ and let $\omega=(\omega_{ij})$ be its connection 1-form, i.e.,

$$
\omega_{ij}=-\frac{(b_i)_{x_j}}{b_j}dx_i+\frac{(b_j)_{x_i}}{b_i}dx_j,\quad i\neq j.
$$

Denote $\delta=\mathrm{diag}(dx_1,\cdots,dx_n)$. Taking $F=(f_{ij})\in gl_*(n)$ with $f_{ij}=\frac{(b_i)_{x_j}}{b_j}$, we obtain $\omega=-(\delta F-F^t\delta)$ and the Gauss equations (2.2) become

$$
d\omega=\omega\wedge\omega-\delta b\wedge b^t\delta.
$$

It is easy to prove that the Codazzi equations (2.3) are equivalent to

$$
\delta\wedge A^{-1}dA=\omega\wedge\delta=-\delta F\delta.
\tag{4.1}
$$

Put $\eta=J\delta F^tJ-F\delta$, then

$$
\delta\wedge\eta=\omega\wedge\delta\quad\text{and thus}\quad\delta\wedge A^{-1}dA=\delta\wedge\eta.
\tag{4.2}
$$

LEMMA 4.1. *$d\eta=A^{-1}dA$ and $d\eta+\eta\wedge\eta=0$.*

Proof. By (4.1), $\delta\wedge d\eta=\delta\wedge A^{-1}dA$ and hence $A^{-1}dA=\eta+\delta D$ for some $D=(d_{ij})\in gl_*(n)$ since $A^{-1}dA,\ \eta\in so(n-1,1)$. Evaluating this at $(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})$ and $(\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_i})$, we have

$$\sum_k \epsilon_i \epsilon_k a_{ki} (a_{kj})_{x_j} = f_{ij} + d_{ij}, \quad i \neq j,$$

$$\sum_k \epsilon_j \epsilon_k a_{ki} (a_{kj})_{x_i} = -f_{ji}, \quad i \neq j.$$

But since $A \in SO(n-1,1)$, $JA^tJA = I$ and thus

$$f_{ij} + d_{ij} = \sum_k \epsilon_i \epsilon_k a_{ki} (a_{kj})_{x_j} = -\sum_k \epsilon_i \epsilon_k a_{ki} (a_{kj})_{x_i} = -(-f_{ij}).$$

Hence $d_{ij} = 0$. $d\eta + \eta \wedge \eta = 0$ follows easily from $d\eta = A^{-1}dA$. □

We summarize the above arguments and the lemma to conclude

Theorem 4.2. *A nondegenerate local immersion X of the Riemannian manifold $\mathcal{O} \subset \mathbb{S}^n$ in $\mathbb{S}^{2n-1,1}$ of constant sectional curvature 1 with the flat normal bundle as in Theorem (2.2) gives rise to a solution (F,b) of the system on $O(2n,1)/(O(n+1) \times O(n-1,1))$.*

In fact, they are related by

$$F = \left(\frac{(b_i)_{x_j}}{b_j} \right), \quad \omega = -(\delta F - F^t \delta) \quad \text{and} \quad A^{-1}dA = \eta = J\delta F^t J - F\delta.$$

Proof. Taking $e_{n+1} = -X$, we can construct a $so(2n-1,1)/(so(n+1) \times so(n-1,1))$-valued flat connection from the Theorem 2.2,

$$\theta = \begin{pmatrix} -\omega & -\delta b^t & \delta A \\ b\delta & 0 & 0 \\ -JA^{-1}\delta & 0 & 0 \end{pmatrix}. \tag{4.3}$$

Take a gauge transformation $g = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ on θ gives $g{*}\theta = \Theta_1$, where Θ_λ is as in (3.1). Therefore we get a flat connection Θ_λ for any $\lambda \in C$ and hence (F,b) is a solution of the $O(2n-1,1)/(O(n+1) \times O(n-1,1))$ system. □

Conversely, we have

THEOREM 4.2. *A solution (F, b) of the system on $O(2n, 1)/(O(n-1, 1) \times O(n+1))$ gives rise to a nondegenerate local immersion X of the Riemannian manifold $\mathcal{O} \subset \mathbb{S}^n$ in $\mathbb{S}^{2n-1,1}$ of constant sectional curvature 1 with the flat normal bundle.*

Proof. We have a flat connection Θ_λ from (F, b). Taking a gauge transformation $h = \begin{pmatrix} A^{-1} & 0 \\ 0 & I \end{pmatrix}$ on Θ_1 gives $h * \Theta_1 = \theta$, where θ is given by (4.3). Let $E = (e_1, \cdots, e_{2n})$ be the trivialization of θ. Put $X = -e_{n+1}$, then $< X, X >= 1$ and thus X lies on $\mathbb{S}^{2n-1,1}$. Now, X gives the required immersion. $\square$

REFERENCES

1. M. Brück, X. Du, J. Park, C. Terng, *The submanifold geometry of real Grassmannian systems, to appear.*
2. E. Cartan, *Sur les varièt̀es de courbure constante d'un espace euclidien ou non-euclidien*, Bull. Soc. Math. France **47** (1920), 125–160.
3. D. Hilbert, *Über Flächen von konstanter Gausscher Krümmung*, Trans. AMS. **2** (1901), 89–99.
4. J. Moore, *Isometric immersions of space forms in space forms*, Pacific Jour. Math. **40** (1972), 157–166.
5. B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, 1983.
6. R. Palais, C. Terng, *Critical Point Theory and Submanifold Geometry*, LNM 1353, 1988.
7. C. Terng, *Soliton equations and differential geometry*, Diff. Geom. **45** (1997), 407–445.

DEPARTMENT OF MATHEMATICS
DONGGUK UNIVERSITY
SEOEL 100-751, KOREA

E-mail: jpark@mail.dgu.ac.kr