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HOPF STRUCTURE FOR POISSON ENVELOPING ALGEBRAS

Kangju Min and Sei-Qwon Oh

Abstract. For a Poisson Hopf algebra A, we find a natural Hopf structure in the Poisson enveloping algebra $U(A)$ of A. As an application, we show that the Poisson enveloping algebra $U(\mathcal{S}(L))$, where $\mathcal{S}(L)$ is the symmetric algebra of a Lie algebra L, has a Hopf structure such that a sub-Hopf algebra of $U(\mathcal{S}(L))$ is Hopf-isomorphic to the universal enveloping algebra of L

Assume throughout that k denotes a field of characteristic zero. Recall that $A = (A, \cdot, \{,\})$ is said to be a Poisson algebra if $(A, \cdot)$ is a commutative k-algebra and $(A, \{,\})$ is a Lie algebra such that

$$\{ab, c\} = a\{b, c\} + b\{a, c\}$$

for all $a, b, c \in A$.

The Poisson enveloping algebra $U(A)$ of A was constructed in [**6**]. The main purpose of this paper is to see that $U(A)$ has a natural Hopf structure if A is a Poisson Hopf algebra. Let L be a Lie algebra over k and let $\mathcal{S}(L)$ be the symmetric algebra of L. Then $\mathcal{S}(L)$ has a natural Poisson structure induced by the Lie algebra L (see [**1**, 2.8.7] or [**2**, Example 1]) and the subspace of homogeneous elements of $\mathcal{S}(L)$ with degree 1 is equal to L. The second aim of this paper is to see that $U(\mathcal{S}(L))$ contains a sub-Hopf algebra isomorphic to the universal enveloping algebra $U(L)$ of L.

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Throughout the paper, every vector space will be over k and every algebra will be an associative k-algebra with unity. For an algebra B, B_L will be the Lie algebra B with Lie bracket $[a,b] = ab - ba$ for all $a, b \in B$.

Let us review the definition for Poisson enveloping algebra (see [**6**, 3]): For a Poisson algebra A, a triple $(U(A), \alpha_A, \beta_A)$, where $U(A)$ is an algebra, $\alpha_A : A \longrightarrow U(A)$ is an algebra homomorphism and $\beta_A : A \longrightarrow U(A)_L$ is a Lie homomorphism such that

$$\alpha(\{a,b\}) = \beta(a)\alpha(b) - \alpha(b)\beta(a) \text{ and } \beta(ab) = \alpha(a)\beta(b) + \alpha(b)\beta(a)$$

for all $a, b \in A$, is called the Poisson enveloping algebra for A if $(U(A), \alpha_A, \beta_A)$ satisfies the following; if B is an algebra, γ is an algebra homomorphism from A into B and δ is a Lie homomorphism from $(A, \{,\})$ into B_L such that

$$\gamma(\{a,b\}) = \delta(a)\gamma(b) - \gamma(b)\delta(a) \text{ and } \delta(ab) = \gamma(a)\delta(b) + \gamma(b)\delta(a)$$

for all $a, b \in A$, then there exists a unique algebra homomorphism h from $U(A)$ into B such that $h\alpha_A = \gamma$ and $h\beta_A = \delta$.

Note that every Poisson algebra A has a unique Poisson enveloping algebra $U(A)$ which is generated by $\alpha_A(A)$ and $\beta_A(A)$ as an algebra and that a k-vector space M is a simple left $U(A)$-module if and only if M is a left Poisson A-module (see [**6**, 1, 5 and 6]). Moreover Im(β), the image of β, is a Lie algebra with Lie bracket

$$[\beta(a), \beta(b)] = \beta(\{a,b\})$$

for all $a, b \in A$.

DEFINITION 2. (see [**3**, 3.1.3]) A Poisson algebra A is said to be a Poisson Hopf algebra if A is also a Hopf algebra $(A, \iota, \mu, \epsilon, \Delta, S)$ over k such that both structures are compatible in the sense that

$$\Delta(\{a, b\}) = \{\Delta(a), \Delta(b)\}_{A\otimes A}$$

for all $a, b \in A$, where the Poisson bracket $\{,\}_{A\otimes A}$ on $A \otimes A$ is defined by

$$\{a \otimes a', b \otimes b'\} = \{a, b\} \otimes a'b' + ab \otimes \{a', b'\}$$

for all $a, a', b, b' \in A$.

For example, every coordinate ring of Poisson Lie group is a Poisson Hopf algebra.

LEMMA 3. *Let A be a Poisson algebra and let $(U(A), \alpha, \beta)$ be the Poisson enveloping algebra for A. Then*

(i) $\alpha\otimes\alpha : A\otimes A \longrightarrow U(A)\otimes U(A)$ *is an algebra homomorphism.*
(ii) $\alpha \otimes \beta + \beta \otimes \alpha : A \otimes A \longrightarrow (U(A) \otimes U(A))_L$ *is a Lie homomorphism.*

Proof. Straightforward. □

Let A and B be Poisson algebras. An algebra homomorphism $\phi : A \longrightarrow B$ is said to be a Poisson homomorphism if ϕ is also a Lie homomorphism.

LEMMA 4. *Given Poisson algebras A, B and an algebra C, let $\phi : A \longrightarrow B$ be a Poisson homomorphism, let $\alpha : B \longrightarrow C$ be an algebra homomorphism and let $\beta : B \longrightarrow C_L$ be a Lie homomorphism such that*

$$\alpha(\{b_1, b_2\}) = \beta(b_1)\alpha(b_2) - \alpha(b_2)\beta(b_1),$$
$$\beta(b_1b_2) = \alpha(b_1)\beta(b_2) + \alpha(b_2)\beta(b_1)$$

for all $b_1, b_2 \in B$. *Then* $\alpha\phi$ *is an algebra homomorphism from* A *into* C *and* $\beta\phi$ *is a Lie homomorphism from* A *into* C_L *such that*

$$(\alpha\phi)(\{a_1, a_2\}) = (\beta\phi)(a_1)(\alpha\phi)(a_2) - (\alpha\phi)(a_2)(\beta\phi)(a_1)$$
$$(\beta\phi)(a_1a_2) = (\alpha\phi)(a_1)(\beta\phi)(a_2) + (\alpha\phi)(a_2)(\beta\phi)(a_1)$$

for all $a_1, a_2 \in A$.

Proof. Straightforward. □

LEMMA 5. *Let* A *be a Poisson algebra and let* $(U(A), \alpha, \beta)$ *be the Poisson enveloping algebra for* A. *Then* $(U(A)\otimes U(A), \alpha\otimes\alpha, \alpha\otimes\beta + \beta\otimes\alpha)$ *is the Poisson enveloping algebra for* $A\otimes A$.

Proof. It is straightforward to see that

$$\begin{aligned}(\alpha\otimes\alpha)(\{a\otimes a', b\otimes b'\}) &= (\alpha\otimes\beta+\beta\otimes\alpha)(a\otimes a')(\alpha\otimes\alpha)(b\otimes b')\\ &\quad - (\alpha\otimes\alpha)(b\otimes b')(\alpha\otimes\beta+\beta\otimes\alpha)(a\otimes a')\end{aligned}$$

$$\begin{aligned}(\alpha\otimes\beta+\beta\otimes\alpha)((a\otimes a')(b\otimes b')) &= (\alpha\otimes\alpha)(a\otimes a')(\alpha\otimes\beta+\beta\otimes\alpha)(b\otimes b')\\ &\quad + (\alpha\otimes\alpha)(b\otimes b')(\alpha\otimes\beta+\beta\otimes\alpha)(a\otimes a').\end{aligned}$$

Let i_1 and i_2 be the Poisson homomorphisms from A into $A\otimes A$ defined by

$$i_1 : A \longrightarrow A\otimes A, \qquad i_1(a) = a\otimes 1$$
$$i_2 : A \longrightarrow A\otimes A, \qquad i_2(a) = 1\otimes a$$

for all $a \in A$. Given an algebra B, let μ_B be the multiplication map on B. If γ is an algebra homomorphism from $A\otimes A$ into B and δ is a Lie homomorphism from $A\otimes A$ into B_L such that

$$\gamma(\{a\otimes a', b\otimes b'\}) = \delta(a\otimes a')\gamma(b\otimes b') - \gamma(b\otimes b')\delta(a\otimes a')$$
$$\delta((a\otimes a')(b\otimes b')) = \gamma(a\otimes a')\delta(b\otimes b') + \gamma(b\otimes b')\delta(a\otimes a')$$

for all $a, a', b, b' \in A$, then there exist algebra homomorphisms f, g from $U(A)$ into B such that $f\alpha = \gamma i_1, f\beta = \delta i_1, g\alpha = \gamma i_2, g\beta = \delta i_2$ by Lemma 4.

$$\begin{array}{ccc} U(A) & \xrightarrow{f} & B \\ \uparrow{\scriptstyle \alpha,\beta} & & \uparrow{\scriptstyle \gamma,\delta} \\ A & \xrightarrow{i_1} & A \otimes A \end{array} \qquad \begin{array}{ccc} U(A) & \xrightarrow{g} & B \\ \uparrow{\scriptstyle \alpha,\beta} & & \uparrow{\scriptstyle \gamma,\delta} \\ A & \xrightarrow{i_2} & A \otimes A \end{array}$$

Hence, we have

$$\begin{aligned} \mu_B(f \otimes g)(\alpha \otimes \alpha)(a \otimes a') &= \gamma i_1(a)\gamma i_2(a') = \gamma(a \otimes a') \\ \mu_B(f \otimes g)(\alpha \otimes \beta + \beta \otimes \alpha)(a \otimes a') &= \gamma i_1(a)\delta i_2(a') + \delta i_1(a)\gamma i_2(a') \\ &= \delta(a \otimes a'). \end{aligned}$$

Thus $\mu_B(f \otimes g)$ is an algebra homomorphism such that

$$\mu_B(f \otimes g)(\alpha \otimes \alpha) = \gamma \quad \text{and} \quad \mu_B(f \otimes g)(\alpha \otimes \beta + \beta \otimes \alpha) = \delta$$

and such an algebra homomorphism $\mu_B(f \otimes g)$ is unique since $U(A)$ is generated by $\alpha(A)$ and $\beta(A)$. □

LEMMA 6. *Let A and B be Poisson algebras and let $(U(A), \alpha_A, \beta_A)$ and $(U(B), \alpha_B, \beta_B)$ be Poisson enveloping algebras for A and B respectively. If $\phi : A \longrightarrow B$ is a Poisson homomorphism then there exists a unique algebra homomorphism $U(\phi) : U(A) \longrightarrow U(B)$ such that $U(\phi)\alpha_A = \alpha_B\phi$ and $U(\phi)\beta_A = \beta_B\phi$.*

$$\begin{array}{ccc} U(A) & \xrightarrow{U(\phi)} & U(B) \\ \uparrow{\scriptstyle \alpha_A,\beta_A} & & \uparrow{\scriptstyle \alpha_B,\beta_B} \\ A & \xrightarrow{\phi} & B \end{array}$$

Proof. It follows immediately from the definition of Poisson enveloping algebra and Lemma 4. □

Let $A = (A, \cdot, \{,\})$ be a Poisson algebra. Define a k-bilinear map $\{,\}_1$ on A by

$$\{a, b\}_1 = \{b, a\}$$

for all $a, b \in A$. Then $A_1 = (A, \cdot, \{,\}_1)$ is a Poisson algebra. For an algebra B, we denote by $B^{\text{op}} = (B, \circ)$ the opposite algebra of B.

PROPOSITION 7. *Let A be a Poisson algebra and let $(U(A), \alpha, \beta)$ be the Poisson enveloping algebra for A. Then $(U(A)^{\text{op}}, \alpha, \beta)$ is the Poisson enveloping algebra for A_1.*

Proof. Clearly, α is an algebra homomorphism from A_1 into $U(A)^{\text{op}}$ and β is a Lie homomorphism from A_1 into $U(A)^{\text{op}}_L$. Moreover, by [**7**, 16], we have

$$\begin{aligned}\alpha(\{a, b\}_1) &= \alpha(\{b, a\}) = \alpha(b)\beta(a) - \beta(a)\alpha(b)\\ &= \beta(a) \circ \alpha(b) - \alpha(b) \circ \beta(a)\end{aligned}$$

$$\beta(ab) = \beta(a)\alpha(b) + \beta(b)\alpha(a) = \alpha(a) \circ \beta(b) + \alpha(b) \circ \beta(a)$$

for all $a, b \in A_1$. If B is an algebra, $\gamma : A_1 \longrightarrow B$ is an algebra homomorphism and $\delta : A_1 \longrightarrow B_L$ is a Lie homomorphism such that

$$\gamma(\{a, b\}_1) = \delta(a)\gamma(b) - \gamma(b)\delta(a) \text{ and } \delta(ab) = \gamma(a)\delta(b) + \gamma(b)\delta(a)$$

for all $a, b \in A_1$, then $\gamma : A \longrightarrow B^{\text{op}}$ is an algebra homomorphism and $\delta : A \longrightarrow B^{\text{op}}_L$ is a Lie homomorphism such that

$$\begin{aligned}\gamma(\{a, b\}) &= \gamma(\{b, a\}_1) = \gamma(b)\delta(a) - \delta(a)\gamma(b)\\ &= \delta(a) \circ \gamma(b) - \gamma(b) \circ \delta(a)\end{aligned}$$

$$\delta(ab) = \delta(a)\gamma(b) + \delta(b)\gamma(a) = \gamma(a) \circ \delta(b) + \gamma(b) \circ \delta(a)$$

for all $a, b \in A$ by [**7**, 16]. Hence there is a unique algebra homomorphism h from $U(A)$ into B^{op} such that $h\alpha = \gamma$ and $h\beta = \delta$ and so $h : U(A)^{\text{op}} \longrightarrow B$ is a unique algebra homomorphism such that $h\alpha = \gamma$ and $h\beta = \delta$. Thus $(U(A)^{\text{op}}, \alpha, \beta)$ is the Poisson enveloping algebra for A_1. □

THEOREM 8. *If* $(A, \iota, \mu, \epsilon, \Delta, S)$ *is a Poisson Hopf algebra then* $(U(A), \iota, \mu, U(\epsilon), U(\Delta), U(S))$ *is a Hopf algebra such that*

$$\begin{aligned} U(\Delta)\alpha &= (\alpha \otimes \alpha)\Delta & \quad U(\Delta)\beta &= (\alpha \otimes \beta + \beta \otimes \alpha)\Delta \\ U(\epsilon)\alpha &= \epsilon & \quad U(\epsilon)\beta &= 0 \\ U(S)\alpha &= \alpha S & \quad U(S)\beta &= \beta S. \end{aligned}$$

Proof. Since Δ is a Poisson homomorphism and $U(A) \otimes U(A)$ is the Poisson enveloping algebra of $A \otimes A$ by Lemma 5, there exists an algebra homomorphism $U(\Delta)$ from $U(A)$ into $U(A) \otimes U(A)$ such that

$$U(\Delta)\alpha = (\alpha \otimes \alpha)\Delta, \quad U(\Delta)\beta = (\alpha \otimes \beta + \beta \otimes \alpha)\Delta$$

by Lemma 6. Similarly, there exists an algebra homomorphism $U(\epsilon)$ from $U(A)$ into k such that $U(\epsilon)\alpha = \epsilon$, $U(\epsilon)\beta = 0$ since $(k, \mathrm{id}_k, 0)$ is the Poisson enveloping algebra of the scalar field k with trivial Poisson bracket. Since A is a commutative algebra, the antipode S is a Poisson homomorphism from A into A_1 and so there is an algebra homomorphism $U(S) : U(A) \longrightarrow U(A)^{\mathrm{op}}$ such that $U(S)\alpha = \alpha S$ and $U(S)\beta = \beta S$ by Lemma 5 and Lemma 6. It is verified routinely that $(U(A), \iota, \mu, U(\epsilon), U(\Delta), U(S))$ is a Hopf algebra. □

Hereafter, we denote by L a Lie algebra with Lie bracket $[-,-]$, by $U(L)$ the universal enveloping algebra of the Lie algebra L and by $\mathcal{S}(L)$ the symmetric algebra of L. Note that

$$\mathcal{S}(L) = k1 \bigoplus L \bigoplus L^2 \bigoplus \cdots$$

as a vector space. Then, by [**1**, 2.8.7] or [**2**, Example 1], $\mathcal{S}(L)$ is a Poisson algebra with Poisson bracket

$$\{a, b\} = [a, b]$$

for all $a, b \in L$. Let $U = (U(\mathcal{S}(L)), \alpha, \beta)$ be the Poisson enveloping algebra of $\mathcal{S}(L)$ and let U' be the subalgebra of U generated by $\beta(L)$. Note that $U(L)$ has the Hopf structure

$$\Delta(a) = a \otimes 1 + 1 \otimes a,\ \ \epsilon(a) = 0,\ \ S(a) = -a$$

for all $a \in L$.

PROPOSITION 9. *$(U', \beta j)$ is the universal enveloping algebra of L, where j is the inclusion map from L into $\mathcal{S}(L)$.*

Proof. Given an algebra B and a Lie homomorphism $f : L \longrightarrow B_L$, define two k-linear maps f', f'' from $\mathcal{S}(L)$ into B by

$$\begin{array}{lll} f'(1) = 1 & f'(L) = 0 & f'(L^i) = 0 \text{ for all } i = 2, 3, \cdots \\ f''(1) = 0 & f''|_L = f & f''(L^i) = 0 \text{ for all } i = 2, 3, \cdots . \end{array}$$

Clearly, $f'(xy) = f'(x)f'(y)$ and $f''(\{x, y\}) = \{f''(x), f''(y)\}$ for all elements $x, y \in \mathcal{S}(L)$. Moreover, f' and f'' satisfy

$$f'(\{x, y\}) = 0 = f''(x)f'(y) - f'(y)f''(x)$$

$$f''(xy) = f'(x)f''(y) + f'(y)f''(x)$$

for all $x, y \in \mathcal{S}(L)$. Hence there exists a unique algebra homomorphism $h : U \longrightarrow B$ such that $h\alpha = f'$ and $h\beta = f''$. Since $h\beta j = f''j = f$ and U' is generated by $\beta j(L)$, the map $h|_{U'}$ is a unique algebra homomorphism such that $(h|_{U'})\beta j = f$, and so $(U', \beta j)$ is the universal enveloping algebra of L. □

THEOREM 10. *Let $U = (U(\mathcal{S}(L)), \alpha, \beta)$ be the Poisson enveloping algebra of $\mathcal{S}(L)$ and let U' be the subalgebra of U generated by $\beta(L)$. Then U has a Hopf structure such that*

$$\begin{array}{ll} \Delta\alpha(a) = \alpha(a) \otimes 1 + 1 \otimes \alpha(a) & \Delta\beta(a) = \beta(a) \otimes 1 + 1 \otimes \beta(a) \\ \epsilon\alpha(a) = 0 & \epsilon\beta(a) = 0 \\ S\alpha(a) = -\alpha(a) & S\beta(a) = -\beta(a) \end{array}$$

for all $a \in L$. Hence U' is Hopf-isomorphic to $U(L)$.

Proof. Note that $\mathcal{S}(L)$ has the Hopf structure

$$\Delta(a) = a \otimes 1 + 1 \otimes a,\ \epsilon(a) = 0,\ S(a) = -a$$

for all $a \in L$. We shall show that $(\mathcal{S}(L), \iota, \mu, \epsilon = 0, \Delta, S)$ is a Poisson Hopf algebra. For this, it is enough to prove the following:

$$\Delta(\{x, y\}) = \{\Delta(x), \Delta(y)\} \tag{1}$$

$$\epsilon(\{x, y\}) = 0 \tag{2}$$

$$S(\{x, y\}) = \{S(y), S(x)\} \tag{3}$$

for all $x, y \in \mathcal{S}(L)$. Clearly (2) is true. We may assume that x and y are homogeneous elements. We proceed by induction on the homogeneous degree. If x or y is of homogeneous element with degree 0 then (1) and (3) are clearly true. If $x, y \in L$ then

$$\begin{aligned}
\Delta(\{x, y\}) &= \{x, y\} \otimes 1 + 1 \otimes \{x, y\} \\
&= \{x \otimes 1 + 1 \otimes x,\ y \otimes 1 + 1 \otimes y\} = \{\Delta(x), \Delta(y)\} \\
S(\{x, y\}) &= -\{x, y\} = \{-y, -x\} = \{S(y), S(x)\}.
\end{aligned}$$

Fix a homogeneous element $y \in L^j$ and assume that (1) and (3) are true for every homogeneous element x with degree $\le i$. For homogeneous elements $a \in L^r, b \in L^s$ such that $r \le i, s \le i, i < r + s$, we have that

$$\begin{aligned}
\Delta(\{ab, y\}) &= \Delta(a\{b, y\} + b\{a, y\}) \\
&= \Delta(a)\Delta(\{b, y\}) + \Delta(b)\Delta(\{a, y\}) \\
&= \Delta(a)\{\Delta(b), \Delta(y)\} + \Delta(b)\{\Delta(a), \Delta(y)\} \\
&= \{\Delta(ab), \Delta(y)\} \\
S(\{ab, y\}) &= S(a\{b, y\} + b\{a, y\}) \\
&= S(a)\{S(y), S(b)\} + S(b)\{S(y), S(a)\} = \{S(y), S(ab)\}
\end{aligned}$$

by the induction hypothesis. Thus (1) and (3) are true for arbitrary homogeneous elements $x \in \mathcal{S}(L)$ and $y \in L^j$. Similarly if $a \in L^r, b \in L^s$ such that $r \le j, s \le j, j < r+s$ then

$$\begin{aligned}
\Delta(\{x, ab\}) &= \Delta(a\{x,b\} + b\{x,a\}) \\
&= \Delta(a)\Delta(\{x,b\}) + \Delta(b)\Delta(\{x,a\}) \\
&= \Delta(a)\{\Delta(x), \Delta(b)\} + \Delta(b)\{\Delta(x), \Delta(a)\} \\
&= \{\Delta(x), \Delta(ab)\} \\
S(\{x, ab\}) &= S(a\{x,b\} + b\{x,a\}) \\
&= S(a)\{S(b), S(x)\} + S(b)\{S(a), S(x)\} \\
&= \{S(ab), S(x)\}.
\end{aligned}$$

Therefore (1) and (3) are true for all homogeneous elements $x, y \in \mathcal{S}(L)$. Hence $\mathcal{S}(L)$ is a Poisson Hopf algebra and so $(U, \iota, \mu, \epsilon = 0, \Delta, S)$ is a Hopf algebra by Theorem 8. Clearly U' is a sub-Hopf algebra isomorphic to $U(L)$ by Proposition 9. □

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Department of Mathematics
Chungnam National University
Taejon 305-764, Korea

E-mail: sqoh@math.chungnam.ac.kr