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ON THE MODERATE DEVIATION TYPE FOR RANDOM AMOUNT OF SOME RANDOM MEASURES

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Abstract. In this paper we study another kind of the large deviation property, i.e. moderate deviation type for random amount of random measures on R^d about a Poisson point process and a Poisson center cluster random measure.

1. Introduction

Suppose thst S is a Polish space and B_s is its Borel σ-algebra. A function $I(\cdot)$ from S into $[0, \infty]$ is called a rate function if $I(\cdot)$ is lower semicontinuous and the level sets $\{x \in S : I(x) < c\}$ are compact sets in S for each real number $c < \infty$.

Let P_ϵ be a family of probability measures on the Borel subsets of S. We say that the family P_ϵ satisfies a large deviation property with a rate function $I(\cdot)$ if

$$\limsup_{\epsilon \to 0} \epsilon \cdot \log P_\epsilon(F) \leq - \inf_{x \in F} I(x)$$

for each closed set F in S and

$$\liminf_{\epsilon \to 0} \epsilon \cdot \log P_\epsilon(G) \geq - \inf_{x \in G} I(x)$$

for each open set G in S.

The simplest situation of large deviations is to take for P_n the distribution on the real line corresponding to the mean $(X_1+X_2+\cdots+X_n)/n$ of n independent identically distributed random variables.

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In this paper we want to study another kind of a large deviation property, i.e. moderate deviation type for random amount of some random measures on R^d about a Poisson point process and a Poisson center cluster random measure.

2. Preliminaries and general results

In this section, we introduce a theorem given by Ellis(1985), which plays a crucial role in this note. We also introduce a theorem given by Hwang(1993).

Let $\{W_n; n = 1, 2, \cdots\}$ be a sequence of random variables which are defined on probability spaces $\{(\Omega_n, F_n, P_n); n = 1, 2, \cdots\}$ and which take values in R. We define function $C_n(t)$ using the cumulant generating function, i.e.,

$$C_n(t) = \frac{1}{a_n} \cdot \log E_n[e^{tW_n}], n = 1, 2, \cdots, t \in R,$$

where $\{a_n; n = 1, 2, \cdots\}$ is a sequence of positive numbers tending to infinity and E_n denotes expectation with respect to P_n. The following hypotheses are assumed to hold.

(i) Each function $C_n(t)$ is finite for every $t \in R$.

(ii) $C(t) = \lim_{n\to\infty} C_n(t)$ exists for every $t \in R$ and is finite. We define the function $I : R \to [0, \infty]$ by the Legendre-Fenchel transform

$$I(x) = \sup_{t\in R}\{tx - C(t)\}, x \in R.$$

THEOREM 2.1. ([3]) *Let P_n be the distribution of W_n/a_n on R. Under the above hypotheses (i) and (ii), the following conclusion hold.*

(a) $I(x)$ is lower semicontinuous, nonnegative and has compact level sets with $\inf_{x\in R} I(x) = 0$

(b) The upper large deviation bound is valid :

$$\limsup_{n\to\infty} \frac{1}{a_n} \cdot \log P_n(F) \le -\inf_{x\in F} I(x) \text{ for each closed set } F \text{ in } R.$$

(c) Moreover, if $C(t)$ is in addition differentiable for all t, then the lower large deviation bound is valid :

$\liminf_{n\to\infty} \frac{1}{a_n} \cdot \log P_n(G) \geq -\inf_{x\in G} I(x)$ *for each open set G in R.*

Hence, if (b) and (c) hold for all closed subsets F and (c) for all open subsets G, respectively, then $\{P_n; n = 1, 2, \cdots\}$ satisfies a large deviation property with respect to $\{a_n; n \geq 1\}$ and with the rate function $I(x)$.

We now assume that the readers are familiar with the language of random measures. As a good reference for details, the reader may consult Kallenberg's book(1983).

Let N be the set of Radon(i.e., locally finite) Borel measures on R^d, so that if $\mu \in N$ then it is finite on bounded Borel sets. Let $\tilde{N}$ be the σ-algebra of subsets of N generated by sets of the form

$$\{\mu \in N : \mu(B) < r\}$$

for a bounded Borel set B and a nonnegative real number r. A random measure is a measurable function X from a fixed probability space (Ω, A, P) into $(N, \tilde{N})$.

If $B \subseteq R^d$ is a Borel subset, then we let $X(B)$ be the random amount of mass the measure X gives B. Similarly, we let $X(f)$ be the integral of $f : R \to R$ with respect to the random measure X if this is defined. We assume X to be stationary(i.e., to have a translation invariant distribution) and ergodic. The most well known random measures are the Poisson point process and the Poisson center cluster random measure.

We define random amount $X_r(B)$ by $X_r(B) = X(rB)$ for $r \in R^+$. Let $X_r(B)/r^d$ be the random variables obtained by rescaling random measures X_r. The ergodic theorem implies that $X_r(B)/r^d$ converges to a mean of $X_r(B)/r^d$ as $r \to \infty$. Hwang showed that $X_r(B)/r^d$ satisfies a large deviation property in the following theorem.

THEOREM 2.2. *Let P_r be the distribution of $X_r(B)/r^d$. Suppose that $\Phi(t) := \lim_{r\to\infty} \frac{1}{r^d} \cdot \log E[e^{tX_r(B)}]$ exists for each $t \in R$ and is finite. We also define the Legendre-Fenchel transform of $\Phi(t)$ by $I(x) = \sup_{t\in R}\{tx - \Phi(t)\}$ for each $x \in R$. Then, the following conclusions hold.*

(a) $I(x)$ is a rate function and $\inf_{x\in R} I(x) = 0$.

(b) For each closed set F in R,

$$\limsup_{r\to\infty} \frac{1}{r^d} \cdot \log P_r\left(\frac{X_r(B)}{r^d} \in F\right) \le -\inf_{x\in F} I(x).$$

(c) If $\Phi(t)$ is in addition differentiable for each $t \in R$, then for each open set G in R,

$$\liminf_{r\to\infty} \frac{1}{r^d} \cdot \log P_r\left(\frac{X_r(B)}{r^d} \in G\right) \ge -\inf_{x\in G} I(x).$$

Hence, if (b) and (c) hold for all closed subsets F and for each open subsets G , respectively, then $\{P_r; r \in R^+\}$ satisfies a large deviation property with a rate function I.

3. Moderate deviation type for some random measures

In this section we consider a Poisson point process and a Poisson center cluster random measure. These are general classes of cluster models which include the Poisson, Neyman-Scott cluster processes and self-exciting processes. For more details the reader may consult Kallenberg [6] and Karr [7].

Let X be a random measure on R^d and let B be a Borel subset of R^d . We recall random amount $X_r(B)$ by $X_r(B) = X(rB)$ for $r \in R^+$. The ergodic theorem implies that $X_r(B)/r^d$ converges to a mean of $X_r(B)/r^d$ as $r \to \infty$. Let P_r be the distribution of $X_r(B)/r^d$, the random variables obtained by rescaling random measures X_r. Hwang investigated a large deviation property for the probability measures P_r with respect to the sequence $\{r^d; r \in R^+, d \ge 1\}$ and with a rate function I depending on the moment generating function.

Here we consider a sequence $\{a_r; r \in R^+\}$ of positive numbers which go to ∞ as $r \to \infty$ and such that $\frac{a_r}{r^{d/2}} \to 0$ as $r \to \infty$. The sequence of interest in moderate deviation type is $T(r) := \{\frac{a_r}{r^{d/2}}\{X_r(B) - E[X_r(B)]\}; r \in R^+\}$ for a Borel subset B of R^d. This result falls into the realm of moderate deviation properties.

Thus, we want to show the moderate deviation type for random amount of some random measures. This means that $T(r)$ satisfies a large deviation property with respect to $\{{a_r}^2; r \in R^+\}$ and with the rate function $I(\cdot)$.

3.1. Poisson point process with intensity $\alpha > 0$. We say that X is a Poisson point process with intensity $\alpha > 0$ if $X(B)$ is a Poisson random variable with parameter $\alpha|B|$ for each bounded Borel set B in R^d and $X(B_1), X(B_2), \cdots, X(B_n)$ are independent Poisson random variables with respective parameters $\alpha|B_1|, \alpha|B_2|, \cdots, \alpha|B_n|$ for disjoint bounded Borel sets $B_1, B_2, \cdots, B_n$ in R^d, where $|\cdot|$ denotes Lebesgue measure.

For a Poisson point process X with intensity $\alpha > 0$, we get the cumulant generating function $C_r(t)$ of the random variables $X_r(B)$, i.e.,

$$C_r(t) = \frac{1}{{a_r}^2} \cdot \log E_r[e^{tX_r(B)}] = \frac{1}{{a_r}^2} \cdot (e^{\alpha r^d |B|(e^t - 1)}), r \in R^+, t \in R.$$

THEOREM 3.1. *Let X e a Poisson point process with intensity $\alpha > 0$ and let B be a bounded Borel subset of R^d. Suppose that $\frac{a_r}{r^{d/2}} \to 0$ as $r \to \infty$ for a sequence $\{a_r; r \in R^+\}$ of positive numbers which go to infinity as $r \to \infty$. Then we have the following properties:*

(a) $\lim_{r\to\infty} C_r(t) = \frac{1}{2}\alpha|B|t^2, t \in R$ *and* $I(x) = \frac{x^2}{2\alpha|B|}, x \in R$.

(b) $\frac{X_r(B) - E[X_r(B)]}{a_r r^{d/2}}$ *satisfies the* $({a_r}^2)$ *- large deviation property with a rate function* $I(x)$.

Proof. First, let us consider a function $C_r(t)$ of the random variables as follows:

$$\begin{aligned}
C_r(t) &= \frac{1}{a_r{}^2}\log E[e^{\frac{a_r}{r^{d/2}}t\{X_r(B)-E[X_r(B)]\}}] \\
&= -\frac{E[X_r(B)]}{a_r r^{d/2}}t + \frac{1}{a_r{}^2}\log E[e^{\frac{a_r}{r^{d/2}}tX_r(B)}] \\
&= -\frac{\alpha r^d|B|}{a_r r^{d/2}}t + \frac{1}{a_r{}^2}\log\left\{e^{\alpha r^d|B|(e^{\frac{a_r}{r^{d/2}}t}-1)}\right\} \\
&= -\frac{\alpha r^d|B|}{a_r r^{d/2}}t + \frac{\alpha r^d|B|}{a_r{}^2}\left\{e^{\frac{a_r}{r^{d/2}}t}-1\right\} \\
&= -\frac{\alpha r^d|B|}{a_r r^{d/2}}t + \frac{\alpha r^d|B|}{a_r{}^2}\sum_{k=1}^{\infty}\left(\frac{a_r}{r^{d/2}}\right)^k\frac{t^k}{k!} \\
&= \frac{\alpha r^d|B|}{a_r{}^2}\sum_{k=2}^{\infty}\left(\frac{a_r}{r^{d/2}}\right)^k\frac{t^k}{k!} \\
&= \frac{\alpha|B|}{2}t^2 + \alpha|B|\sum_{k=3}^{\infty}\left(\frac{a_r}{r^{d/2}}\right)^{k-2}\frac{t^k}{k!}.
\end{aligned}$$

Since $\frac{a_r}{r^{d/2}} \to 0$ as $r \to \infty$, the second term in the last line converges to $\frac{\alpha|B|}{2}t^2$ as $r \to \infty$. Differentiating $tx - C(t)$ with respect to t, $I(x)$ can be obtained by $I(x) = \frac{x^2}{2\alpha|B|}, x \in R$. For the proof of (b), Theorem 2.1 implies (b) with $a_r = a_r{}^2$ and $W_r = T(r)$. □

3.2. Poisson center cluster random measure. From now on, we follow Burton and Dehling(1990) for terminology. Let U be a stationary Poisson process on R^d with intensity $\alpha > 0$. V is a point process so that $E[V(R^d)] = \zeta < \infty$. We also assume that $V(R^d)$ has a finite moment generating function $M_{V(R^d)}(t) = E[e^{tV(R^d)}]$ for $t \in R$. Let x_i be the random occurrences of U and let V_i be independent identically distributed(i.i.d.) copies of V that are also independent of U. The resulting cluster process X is said to be a Poisson center cluster process, which is defined by superimposing i.i.d. copies of V centered at the occurrences of U.

If B is a bounded Borel subset of R^d, then X is defined by

$$X(B) = \sum_i V_i(B - x_i).$$

Note that $E[X(B)] = \alpha\zeta|B|$.

The following theorem is known as Campbell's formula.

THEOREM 3.2. *The moment generating function of $X(B)$ is*

$$M_{X(B)}(t) = \exp\left\{\alpha \cdot \int_{R^d} E[e^{tV(B-x)} - 1]dx\right\}, t \in R.$$

THEOREM 3.3. *Let X be a Poisson center cluster random measure on R^d and let B be a bounded Borel subset of R^d. We assume that $V(R^d)$ has a finite moment generating function $M_{V(R^d)}(t) = E[e^{tV(R^d)}]$ for $t \in R$. Then we have the following properties:*

(a) $\lim_{r\to\inf} C_r(t) = \frac{1}{2}\alpha|B|t^2, t \in R$ *and* $I(x) = \frac{x^2}{2\alpha|B|E[(V(R^d)^2]}, x \in R.$

(b) $\frac{X_r(B)-E[X_r(B)]}{a_r r^{d/2}}$ *satisfies the* $({a_r}^2)$*- large deviation property with a rate function* $I(x)$.

Proof. First, let us consider some function $C_r(t)$ of the random variables as follows:

$$\begin{aligned}
C_r(t) &= \frac{1}{{a_r}^2}\log E[e^{\frac{a_r}{r^{d/2}}t\{X_r(B)-E[X_r(B)]\}}] \\
&= -\frac{\alpha\zeta r^d|B|}{a_r r^{d/2}}t + \frac{1}{{a_r}^2}\log E[e^{\frac{a_r}{r^{d/2}}tX_r(B)}] \\
&= -\frac{\alpha\zeta r^d|B|}{a_r r^{d/2}}t + \frac{\alpha}{{a_r}^2}\int_{R^d} E[e^{\frac{a_r}{r^{d/2}}tV(rB-x)} - 1]dx \text{ by Theorem 3.2} \\
&= -\frac{\alpha\zeta r^d|B|}{a_r r^{d/2}}t + \frac{\alpha}{{a_r}^2}\sum_{k=1}^{\infty}\left(\frac{a_r}{r^{d/2}}\right)^k\frac{t^k}{k!}\int_{R^d} E[V^k(rB-x)]dx \\
&= -\frac{\alpha\zeta r^d|B|}{a_r r^{d/2}}t + \frac{\alpha r^d}{{a_r}^2}\sum_{k=1}^{\infty}\left(\frac{a_r}{r^{d/2}}\right)^k\frac{t^k}{k!}\cdot \\
&\quad E\left[\int_{R^d}\int_{R^d}\cdots\int_{R^d} 1_{rB-ry}(u_1)\cdots 1_{rB-ry}(u_k)V(du_1)\cdots V(du_k)dy\right] \\
&= -\frac{\alpha\zeta r^d|B|}{a_r r^{d/2}}t + \frac{\alpha r^d}{{a_r}^2}\sum_{k=1}^{\infty}\left(\frac{a_r}{r^{d/2}}\right)^k\frac{t^k}{k!}\cdot \\
&\quad E\left[\int_{R^d}\cdots\int_{R^d} |(B-u_1/r)\cdots(B-u_k/r)|V(du_1)\cdots V(du_k)\right] \\
&= -\frac{\alpha\zeta r^{d/2}|B|}{a_r}t + \frac{\alpha r^{d/2}}{a_r}tE\left[\int_{R^d}|(B-u_1/r)|V(du_1)\right] \\
&\quad +\frac{\alpha}{2}t^2E\left[\int_{R^d}\int_{R^d}|(B-u_1/r)(B-u_2/r)|V(du_1)V(du_2)\right] \\
&\quad +\alpha\sum_{k=3}^{\infty}\left(\frac{a_r}{r^{d/2}}\right)^{k-2}\frac{t^k}{k!} \\
&\quad E\left[\int_{R^d}\cdots\int_{R^d}|(B-u_1/r)\cdots(B-u_k/r)|V(du_1)\cdots V(du_k)\right].
\end{aligned}$$

Since $1_{(B-u_1/r)\cdots(B-u_k/r)} \le 1_{(B-u_1/r)}$ and $|(B-u_1/r)\cdots(B-u_k/r)| \le |(B-u_1/r)| = |B|$, let us apply the Fubini's theorem and the dominated convergence theorem in $C_r(t)$. Then the first term and the second term make zero together, and the third term converges to $\frac{1}{2}\alpha|B|E[(V(R^d))^2]t^2$, and the fourth term converges to zero since $\frac{a_r}{r^{d/2}} \to 0$ as $r \to \infty$. Thus we get that $C(t) = \frac{1}{2}\alpha|B|E[(V(R^d))^2]t^2$ for each $t \in R$. $I(x)$ and (b) can be proved in the same way as Theorem3.1. □

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