JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 13, No.2, December 2000

MININJECTIVE RINGS AND QUASI FROBENIUS RINGS

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Abstract. A ring R is called right mininjective if every isomorphsim between simple right ideals is given by left multiplication by an element of R. In this paper we consider that the necessary and sufficient condition for that Trivial extension of R by V, i.e. $T(R, V)$ is mininjective. We also study the split null extension R and S by V.

A ring R is called quasi Frobenius if it is left and right artinian and left and right selfinjective ; equivalently if R has the ACC on right and left annihilators and is right or left selfinjective. The class of quasi Frobenius rings is one of the most interesting classes of non-semisimple rings and it has been intensively investigated by many authors. Throughout this paper all rings have unity and all modules are unitary. The right and left annihilators of a subset X of a ring R are denoted $r(X)$ and $l(X)$ respectively. We write $J = J(R)$ for the Jacobson radical of R and also $P = P(R)$ for the prime radical of R. If R is a ring, a right module M_R is called mininjective if for each simple right ideal K of R, every R-morphism $\gamma : K \to M$ extends to R ; equivalently if $\gamma = m\cdot$ is left multiplication by some element m of M. Mininjective left modules are defined similarly. Clearly every injective module is mininjective.

These rings were first introduced by Harada[2] who studied the artinian case in [2] and [3]. We begin with several characterizations.

Received by the editors on November 27, 2000.
1991 *Mathematics Subject Classifications*: 16D90, 16P60.
Key words and phrases: mininjective, trivial extension.

Proposition 1. *The following conditions are equivalent for a ring R :*

(1) *R is right mininjective.*
(2) *If kR is simple, $k \in R$, then $lr(k) = Rk$.*
(3) *If kR is simple and $r(k) \subseteq r(a)$, $k, a \in R$, then $Ra \subseteq Rk$.*
(4) *If kR is simple and $\gamma : kR \to R$ is R-linear, then $\gamma(k) \in Rk$.*

Proof. (1) $\Rightarrow$ (2) Given (1), let $0 \neq a \in lr(k)$. $kr = 0$ implies $r \in r(k)$ and $ar = 0$. $\gamma : kR \to R$ given by $\gamma(kr) = ar$ is a well defined homomorphism. Since R is right mininjective, $\gamma(k) = c \cdot k = a$. This implies $a \in Rk$. Clearly $Rk \subseteq lr(k)$. Thus $lr(k) = Rk$.

(2) $\Rightarrow$ (3) If $r(k) \subseteq r(a)$, $k, a \in R$, then $lr(k) \supseteq lr(a)$. This $Rk \supseteq lr(a) \supseteq Ra$.

(3) $\Rightarrow$ (4) If kR is simple and $\gamma : kR \to R$ is R-linear. Let $\gamma(k) = a$. Then $kt = 0$ implies $at = 0$. $r(k) \subseteq r(a) \Rightarrow Ra \subseteq Rk$. Thus $a \in Rk$ and $\gamma(k) \in Rk$.

(4) $\Rightarrow$ (1) Let $\gamma(k) \in Rk$ where kR is simple. Let $\gamma(k) = ck$. If $\varphi : kR \to R$ is a homomorphism, define $\bar{\varphi} : R \to R$ by $\bar{\varphi}(k) = \varphi(k) = ck$. $\bar{\varphi}(r) = cr$ and $\bar{\varphi}$ extends φ. □

A ring is called right principally injective if each R-homomorphsim $aR \to R$, $a \in R$, extends to $R_R \to R_R$. Clearly, every such ring (and hence every right selfinjective ring) is right mininjective.

If all minimal right ideals of a ring R are summands (for example, if R has zero right socle), then R is right mininjective.

Remark 1. *Every polynomial ring $R[x]$ is mininjective because both socles are zero.*

Indeed if $K = kR[x]$ is simple where $deg(k) = n$, then $K = Kx^{n+1}$. So $k \in kR[x]x^{n+1}$ and $deg(k) \geq n + 1$, a contradiction.

REMARK 2. *The ring $\mathbb{Z}$ of integers is a commutative, noetherian, mininjective ring that is not artinian nor principally injective.*

Indeed $soc\mathbb{Z} = 0$.

The right and the left socles of R are denoted $S_r = Soc(R_R)$ and $S_l = Soc({}_RR)$ respectively.

REMARK 3. *If S_r is simple as a left ideal, then R is right mininjective. In fact, if kR is simple and $r(k) \subseteq r(a)$, $k, a \in R, a \neq 0$, then $r(k) = r(a)$ because $r(k)$ is maximal, so Ra is simple too. Hence $Ra = S_r = Rk$ by hypothesis, and apply Proposition 1.*

REMARK 4. *A commutative local ring R is mininjective if and only if $Soc(R)$ is simple or zero.*

For if $Soc(R) \neq 0$, let K and M be simple ideals. Then $K \cong M$ because R is local, so $M = cK$ for $c \in R$ because R is mininjective. Hence $M = K$ and $Soc(R)$ is simple. The converse is by Remark 3.

Hence a commutative artinian ring is quasi Frobenius if and only if it is mininjective.

REMARK 5. *A direct product $\prod_{i\in I} R_i$ of rings R_i right mininjective if and only if R_i is right mininjective for each $i \in I$.*

REMARK 6. *Rutter[6] has an example of a two-sided artinian, right principally injective ring which is not left mininjective.*

REMARK 7. *Camillo[1] has an example of a commutative, local, semiprimary ring with $J^3 = 0$ which is not artinian.*

The next result is half of the proof that mininjectivity is a Morita invariant.

Proposition 2. *If R is right mininjective, so is eRe for all $e^2 = e \in R$ satisfying $ReR = R$.*

Proof. Write $S = eRe$ and let $r_S(k) \subseteq r_S(a)$ where $k, a \in S$ and kS is a simple right ideal of S. We claim first that kR is simple in R. For if $kr \neq 0, r \in R$, then $krReR \neq 0$. So there is $b \in R$ such that $0 \neq krbe = ke(ere) \in kS$. Hence $k = krbeS \subseteq krR$ whence kR is simple. Thus it suffices to show that $r_R(k) \subseteq r_R(a)$ (Then $a \in Rk$ by proposition 1, so $a = ea \in Sk$.) So let $kr = 0, r \in R$, and write $1 = \sum_{i=1}^{n} a_i e b_i$, where $a_i, b_i \in R$. Then $k(era_ie) = kra_ie = 0$ for each i. So $a(ra_ie) = 0$ by hypothesis. Hence $ar = \sum_{i=1}^{n} ara_ieb_i = 0$ as required. □

Proposition 3. *Call a simple right ideal K of R "nice" if maps $K \to R_R$ can be extended to $R \to R$ if $dK \neq 0$ is "nice", $d \in R$, then K is "nice".*

Proof. The map $\sigma(x) = dx$ defines an isomorphism, $\sigma : K \to dK$. Given $\gamma : K \to R_R$, we have $\gamma \circ \sigma^{-1} : dK \to R$ so $\gamma \circ \sigma^{-1} = c\cdot$, $c \in R$ by hypothesis, and it follows that $\gamma = cd\cdot$. □

Proposition 4. *A ring R is right mininjective if and only if $M_n(R)$ is right mininjective for all (some) $n \geq 1$ where $M_n(R)$ is the ring of all $n \times n$ matrices over R.*

Proof. If $S = M_n(R)$ is right mininjective, so is $R \cong e_{11}Se_{11}$ by hyopothesis 1.2, because $Se_{11}S = S$ (here e_{ij} denotes the matrix unit). Coversely if R is right mininjective, let $\bar{k}S$ be a simple right ideal of S. If row of i of $\bar{k}$ is nonzero, then $e_{1i}\bar{k} \neq 0$, so by proposition 3, we may assume that $\bar{k} \in e_{11}S$. Again, if colum j of $\bar{k}$ is nonzero, then $\bar{k}e_{j1} \neq 0$ so $\bar{k}S = \bar{k}e_{j1}S$. Thus we may assume that $\bar{k} \in e_{11}Se_{11}$, so write $\bar{k} = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}$ is block form, $k \in R$. The kR is simple so $Rk = lr(k)$ by Proposition 1. But then

$$r_S(\bar{k}) = \begin{bmatrix} r(k) & r(k) & \cdots & r(k) \\ k & k & \cdots & k \\ \vdots & \vdots & \ddots & \vdots \\ k & k & \cdots & k \end{bmatrix}$$

whence

$$lr_S(\bar{k}) = \begin{bmatrix} lr(k) & 0 & \cdots & 0 \\ lr(k) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ lr(k) & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} Rk & 0 & \cdots & 0 \\ Rk & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Rk & 0 & \cdots & 0 \end{bmatrix}$$
$$= S\bar{k}$$

as required. □

While principal injectivity is not a Morita invariant, combining Proposition 1 and 4 gives.

THEOREM 5. *Right mininjectivity is a Morita invariant.*

Trivial extensions are defined as follows. Given a ring R and a bimodule ${}_RV_R$, the trivial extension of R by V is the ring $S = T(R, V) = R \bigoplus V$ with the usual addition and multiplication

$$(r + v)(r' + v') = rr' + (rv' + vr') \ .$$

This is isomorphic to the ring of all matrices $\begin{bmatrix} r & v \\ 0 & r \end{bmatrix}$ where $r \in R$ and $v \in V$ and the usual matrix operations are used.

PROPOSITION 6. *Let $S = T(R, V)$. Then the followings hold :*

(1) *Every prime ideal P of S has the form $P_1 \bigoplus V$, where P_1 is a prime ideal of R.*

(2) *Every primitive ideal P of S has the form $P_1 \bigoplus V$, where P_1 is a primitive ideal of R.*

(3) *Every maximal ideal M of S has the form $M_1 \bigoplus V$, where M_1 is a maximal ideal of R.*

Proof. $\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $V^2 = 0$. $S/V \cong R$. From these (1), (2) and (3) follow. □

Corollary 7. *Let $S = T(R, V)$. Then the followings hold :*

(1) *The prime radical of $S = P(R) \bigoplus V$.*

(2) *The Jacobson radical of $S = J(R) \bigoplus V$.*

(3) *Every Brown McCary radical of $S = B(R) \bigoplus V$.*

Proposition 8. *If $S = T(R, V)$, then*

$$Soc(S_S) = [Soc(R_R) \quad \cap \quad l_R(V)] \bigoplus Soc(V_R).$$

Proof. For convenience, write $K = Soc(R_R) \cap l_R(V)$. If xS is simple, $x = r_0 + v_0$, then $xV = xSV = 0$ since $V \subseteq J(S)$. Hence $r_0 \in l_R(V)$ and so $xS = xR$. It follows that r_0R and v_0R are simple, so $Soc(S_S) \subseteq K \bigoplus Soc(V_R)$.

Conversely, if v_0R is simple, then $v_0R = v_0S$ is a simple right ideal of S, so $v_0 \in Soc(S_S)$. To show that $K \subseteq Soc(S_S)$, it suffices to show that kS is simple for each $k \in K$ such that kR is simple(K is R-semi-simple). If $0 \neq ks \in kS$, write $x = r_0 + v_0$. Since $kV = 0$, we have $0 \neq kx = kr_0$, so $kr_0R = kR$. Thus $k = kr_0r_1 = krr_1 \in kxS$ as required. □

THEOREM 9. *$S = T(R, V)$ where R is a ring and V is an ideal of R such that $S_r \subset V$ and $l_R(V) = 0$.*

(1) *S is right mininjective if and only if R is right mininjective.*
(2) *S is right artinian if and only if R is right artinian.*

Proof. (1) If S is right mininjective and $\gamma : K \to R_R$ is R-linear, where K is a simple right ideal of R, then $0 + K$ is a simple right ideal of S and we define $\bar{\gamma} : 0 + K \to S$ by $\bar{\gamma}(0 + k) = 0 + \gamma(k)$. $\bar{\gamma}((0+k)(r+v)) = \bar{\gamma}(0+kr) = 0+\gamma(k)r = (0+\gamma(k))(r+v)$. This is S-linear, so $\bar{\gamma}(c+v)\cdot$ by hypothesis. It follows that $\gamma = c\cdot$, proving that R is right mininjective.

Conversely, if R is right mininejctive, let $\gamma : T \to S$ be S-linear, where T is a simple right ideal of S. T and $\gamma(T)$ are contained in $Soc(S_S) = 0 + Soc(R_R)$ by Proposition 7. Define $\gamma_0 : vR \to R$ by $\gamma_0(x) = y$ where $\gamma(0 + x) = 0 + y$. Then γ_0 is R-linear, so $\gamma_0 = c_0$ by hypothesis, where $\gamma = (c_0 + 0)\cdot$.

(2) If S is right artinian, the same is true of $R \cong S/(0 + V)$. Conversely, if R_R is arinian, then S_R is artinian because $S_R \cong R \bigoplus V$ as R-modules. But right ideals of S are right R-modules, and it follows that S_S is artinian. □

THEOREM 10. *Let $S = T(R, V)$, where R is a ring and V is an ideal of R such that $S_l = S_r \subset V \subset R$. Then $l_R(V) = 0$ and $r_R(V) = 0$.*

S is quasi Frobenius if and only if R is quasi Frobenius.

Proof. [4] shows that a ring is quasi-Frobenius if and only if it is right artinian and right and left mininjective. By this and Theorem 8, the statement follows. □

If R and S are rings and ${}_RV_S$ is a bimodule, the split null extensions of R and S by V is the ring $U = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ of all "matrices" $\begin{bmatrix} r & v \\ 0 & s \end{bmatrix}$ using matrix operations.

PROPOSITION 11. *Assume that* $U = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ *is right mininjective. Then*

(1) S *is right mininjective.*
(2) *If* $\gamma : K \to V_S$ *where* $K \subseteq V_S$ *and* K *is a simple* S*-submodule of* S*, then* $\gamma = c\cdot$*, where* $c \in R$.

Proof. (1) If $\gamma : K \to S$, where K is a simple right ideal of S, then $\bar{T} = \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix}$ is a simple right ideal of V. $\bar{\gamma} : \bar{T} \to V$ is S-linear if we define

$$\bar{\gamma} \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \gamma(t) \end{bmatrix} \cdot \quad .$$

Hence $\bar{\gamma} = \begin{bmatrix} r & v \\ 0 & s \end{bmatrix} \cdot$, whence $\gamma = s\cdot$.

(2) If $\gamma : K \to U_S$ where K is a simple submodule of U_S. Then $\bar{\gamma} : \bar{K} \to U$ is V-linear if we define

$$\bar{\gamma} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \gamma(x) \\ 0 & 0 \end{bmatrix} \cdot \quad .$$

Hence $\bar{\gamma} = \begin{bmatrix} c & v \\ 0 & d \end{bmatrix} \cdot$, whence $\gamma = c\cdot$. □

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