JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **13**, No.2, December 2000

HOMOTOPICALLY PERIODIC MAPS OF 3-MANIFOLDS WITH $\widetilde{PSL(2,\mathbb{R})}$-GEOMETRY

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Abstract. In this paper, we show that for a homotopically periodic (i.e., $f^k \simeq id$, for some $k \geq 1$) self map f of a Seifert 3-manifold with $\widetilde{PSL(2,\mathbb{R})}$-geometry, $N(f) = L(f) = 0$.

1. Introduction

If a closed 3-manifold M is a Seifert fiber space, then M is expressed by a quotient X/G, where X is one of $S^2 \times \mathbb{R}, S^3, \mathbb{R}^3, Nil, \mathbf{H}^2 \times \mathbb{R}$ or $\widetilde{PSL(2,\mathbb{R})}$ and G is a discrete cocompact subgroup of the isometry group $Isom(X)$ acting freely on X [**?**]. The appropriate geometry for M is determined by the Euler characteristic χ of the base orbifold and the Euler number e of the Seifert bundle. The determining factors are whether χ is positive, zero or negative and whether e is zero or not. In this paper, we deal with the closed 3-manifolds modeled on $\widetilde{PSL(2,\mathbb{R})}$.

Consider a continuous self map $f : M \to M$ on a smooth closed manifold M. The fixed point set $Fixf = \{x \in M | f(x) = x\}$ is a compact subset of M and is partitioned into finitely many fixed point classes. If the Lefschetz number $L(f)$ is non-vanishing then every map homotopic to f has a fixed point. The Nielsen number $N(f)$ is a lower bound for the number of fixed points of every map in the homotopy

Received by the editors on December 2, 2000.
1991 *Mathematics Subject Classifications* : Primary 55M20, Secondary 55M35.
Key words and phrases: Lefschetz number, Nielsen number $\widetilde{PSL(2,\mathbb{R})}$.

class of f. In fact, if $dimM \geq 3$, then there is a map homotopic to f with exactly $N(f)$ fixed points.

In [**?**], Brooks, Brown, Pak and Taylor proved that for a self map $f : M \to M$ on a torus, the Nielsen number $N(f)$ and Lefschetz number $L(f)$ are equal up to a sign, i.e., $N(f) = |L(f)|$. In [**?**], this result is extended to compact nilmanifolds.

In a series of papers, Kwasik and Lee [**?**] show that $N(f) = L(f)$ for homotopically periodic maps on infranilmanifolds. McCord studied the Nielsen numbers by comparing with the Lefschetz numbers which is readily computable: $|L(f)| \leq N(f)$ for a continuous self map f on a compact solvmanifold [**?**] and on a compact infrasolvmanifold [**?**].

The purpose of this paper is to show that for a homotopically periodic self map f of a Seifert 3-manifold modeled on $\widetilde{PSL(2,\mathbb{R})}$, $N(f) = L(f) = 0$.

2. Results

Let M be a closed 3-manifold. A continuous self map $f : M \to M$ is called *homotopically periodic* if there exists an integer $k \geq 1$ such that f^k is homotopic to the identity.

The 3-dimensional Lie group of all 2×2 real matrices with determinant 1 is denoted $SL(2,\mathbb{R})$, and $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{I, -I\}$, where I is the identity matrix. The unit tangent bundle $U\mathbf{H}^2$ of the hyperbolic plane $\mathbf{H}^2$ is naturally identified with $PSL(2,\mathbb{R})$, the orientation preserving isometry group of $\mathbf{H}^2$. As $U\mathbf{H}^2$ is a circle bundle over $\mathbf{H}^2$, the universal covering $\widetilde{PSL(2,\mathbb{R})}$ has a line bundle structure

$$\mathbb{R} \to \widetilde{PSL(2,\mathbb{R})} \to \mathbf{H}^2$$

over $\mathbf{H}^2$ ($\chi < 0$). This bundle is non-trivial ($e \neq 0$) in a metric sence. That is, $\widetilde{PSL(2,\mathbb{R})}$ is topologically homeomorphic to $\mathbf{H}^2 \times \mathbb{R}$ but not isometric to it.

The isometry group

$$Isom(\widetilde{PSL(2,\mathbb{R})}) = (\mathbb{R} \times_{\mathbb{Z}} \widetilde{PSL(2,\mathbb{R})}) \rtimes \mathbb{Z}_2$$

preserves bundle structure and has only two components. Isometries in the identity component $Isom_o(\widetilde{PSL(2,\mathbb{R})}) = \mathbb{R} \times_{\mathbb{Z}} \widetilde{PSL(2,\mathbb{R})}$ preserve the orientations of the base $\mathbf{H}^2$ and fiber $\mathbb{R}$. The other isometries reverse the orientations of both base and fiber. Thus all isometries of $\widetilde{PSL(2,\mathbb{R})}$ are orientation preserving.

The action of $\mathbb{R}$ on $\widetilde{PSL(2,\mathbb{R})}$ preserves the line bundle structure and covers the identity map of $\mathbf{H}^2$. This $\mathbb{R}$ -action on $\widetilde{PSL(2,\mathbb{R})}$ commutes with the action of $\widetilde{PSL(2,\mathbb{R})}$ and intersects $\widetilde{PSL(2,\mathbb{R})}$ precisely in the center of $\widetilde{PSL(2,\mathbb{R})}$, where the center of $\widetilde{PSL(2,\mathbb{R})}$ is infinite cyclic. So we have the exact sequence

$$1 \to \mathbb{R} \to Isom_o(\widetilde{PSL(2,\mathbb{R})}) \to PSL(2,\mathbb{R}) \to 1.$$

Here $PSL(2,\mathbb{R})$ is the identity component $Isom_o(\mathbf{H}^2)$ of the isometry group $Isom(\mathbf{H}^2)$.

A closed 3-dimensional manifold M modeled on $\widetilde{PSL(2,\mathbb{R})}$ is a quotient of $\widetilde{PSL(2,\mathbb{R})}$ by $\pi = \pi_1(M)$, where π is a cocompact discrete subgroup of $Isom_o(\widetilde{PSL(2,\mathbb{R})})$. Since $\pi \cap \mathbb{R} = \mathbb{Z}$, we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z} & \to & \pi & \to & Q & \to & 1 \\ & & \cap & & \cap & & \cap & & \\ 1 & \to & \mathbb{R} & \to & Isom_o(\widetilde{PSL(2,\mathbb{R})}) & \to & PSL(2,\mathbb{R}) & \to & 1 \end{array}$$

Q is a cocompact discrete orientation-preserving subgroup of $Isom_o(\mathbf{H}^2) = PSL(2,\mathbb{R})$. $\mathbb{Z}$ is the center of π since Q is centerless.

We need the following theorem proved in [**?**].

THEOREM 2.1. *Let $\rho : Q \to PSL(2,\mathbb{R})$ be a discrete cocompact group. For an extension $1 \to \mathbb{Z} \to \pi \to Q \to 1$, there exists a homomorphism $\theta : \pi \to Isom(\widetilde{PSL(2,\mathbb{R})})$ so that the diagram*

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z} & \to & \pi & \to & Q & \to & 1 \\ & & \downarrow & & \downarrow \theta & & \downarrow \rho & & \\ 1 & \to & \mathbb{R} & \to & Isom_o(\widetilde{PSL(2,\mathbb{R})}) & \to & PSL(2,\mathbb{R}) & \to & 1 \end{array}$$

is commutative if and only if $[\pi] \in H^2(Q;\mathbb{Z})$ has infinite order.

Let $\varepsilon(M)$ be the H-space of all self-homotopy equivalences of M into itself. Any $f \in \varepsilon(M)$ induces an isomorphism $f_*([\alpha]) = [\omega^{-1}\cdot(f\circ\alpha)\cdot\omega]$ where ω is a path from x to $f(x)$. Thus we obtain a homomorphism

$$\gamma : \varepsilon(M) \longrightarrow Out(\pi),$$

$\pi = \pi_1(M,x)$. If M is a $K(\pi,1)$ space, then the kernel of γ is $\varepsilon_0(M)$, which is the self homotopy equivalences homotopic to the identity. Since γ is onto, $\pi_0(\varepsilon(M)) = \varepsilon(M)/\varepsilon_0(M)$ is isomorphic to $Out(\pi)$, where $\pi_0(\varepsilon(M))$ is the group of homotopy classes of self-homotopy equvalences.

THEOREM 2.2. *Let M be a 3-dimensional Seifert manifold with geometric structure modeled on $\widetilde{PSL(2,\mathbb{R})}$. If $f : M \to M$ is homotopically periodic, then there is an isometry on M which is homotopic to f.*

Proof. Since f^{k-1} is the homotopy inverse of f, where k is the homotopy period of f, $f \in \varepsilon(M)$. The image of f under the natural map

$$\varepsilon(M) \longrightarrow \pi_0(\varepsilon(M))$$

generates a subgroup $\mathbb{Z}_k = \langle [f] \rangle$ of $\pi_0(\varepsilon(M))$. Since M is a $K(\pi,1)$, $\mathbb{Z}_k$ induces an abstract kernel

$$\varphi : \mathbb{Z}_k \to Out(\pi) \cong \pi_0(\varepsilon(M)).$$

In order to realize $\mathbb{Z}_k$ as a group action on M, it is necessary for the abstract kernel to have an extension

$$1 \to \pi \to E \to \mathbb{Z}_k \to 1$$

[**?**]. Such an extension exists if and only if a certain obstruction class in $H^3(\mathbb{Z}_k; Z(\pi))$ vanishes, where $Z(\pi)$ is the center of π. It is known that $z(\pi) = \mathbb{Z}$ and $H^3(\mathbb{Z}_k;\mathbb{Z})$ is a finite group. Furthermore, the order of $H^3(\mathbb{Z}_k;\mathbb{Z})$ divides k. The short exact sequenc

$$1 \to \mathbb{Z} \xrightarrow{\rho} \mathbb{Z} \to \mathbb{Z}_k \to 1$$

leads to a long exact sequence

$$\to H^2(\mathbb{Z}_k;\mathbb{Z}) \to H^2(\mathbb{Z}_k;\mathbb{Z}_k) \xrightarrow{\delta} H^3(\mathbb{Z}_k;\mathbb{Z}) \to H^3(\mathbb{Z}_k;\mathbb{Z}) \to$$

of cohomology groups. Since the map $H^3(\mathbb{Z}_k;\mathbb{Z}) \to H^3(\mathbb{Z}_k;\mathbb{Z})$ factors through 0, the Bochstein homomorphism δ is onto. Let $x \in H^3(\mathbb{Z}_k;\mathbb{Z})$ be the obstruction for an algebraic realization of the abstract kernel φ. Let $y \in H^2(\mathbb{Z}_k;\mathbb{Z}_k)$ be an element such that $\delta(y) = x$. Then y represents an extension, let us say,

$$1 \to \mathbb{Z}_k \to F \xrightarrow{\rho} \mathbb{Z}_k \to 1.$$

Then the obstruction of new abstract kernel $\phi : F \to Out(\pi)$, where $\phi = \varphi\rho$, is equal to zero in $H^3(F;\mathbb{Z})$. This yields a short exact sequence

$$1 \to \pi \to E \to F \to 1.$$

A characteristic subgroup $\mathbb{Z}$ of π is normal in E. Thus we obtain an exact commutative diagram

$$\begin{array}{ccccccccc} & & \mathbb{Z} & = & \mathbb{Z} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \pi & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & Q & \longrightarrow & Q' & \longrightarrow & F & \longrightarrow & 1, \end{array}$$

where F is finite and $Q \subset Isom_o(\mathbf{H}^2)$. By [**?**] and [**?**], there is a homomorphism

$$Q' \to PSL(2, \mathbb{R}).$$

Since the class $[E] \in H^2(Q'; \mathbb{Z})$ maps to $[\pi] \in H^2(Q; \mathbb{Z})$ under the long exact sequence

$$\to H^2(F; \mathbb{Z}) \to H^2(Q'; \mathbb{Z}) \to H^2(Q; \mathbb{Z}) \to H^1(F; \mathbb{Z}) \to$$

and $[\pi] \in H^2(Q; \mathbb{Z})$ has infinite order, $[E] \in H^2(Q'; \mathbb{Z})$ has infinite order. From the above theorem there exists a homomorphism $E \to Isom_o(\widetilde{PSL(2, \mathbb{R})})$ so that the diagram

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z} & \to & E & \to & Q' & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathbb{R} & \to & Isom_o(\widetilde{PSL(2, \mathbb{R})}) & \to & PSL(2, \mathbb{R}) & \to & 1 \end{array}$$

commutes. Let $g \in E$ maps a generator $[f]$ of $\mathbb{Z}_k$ under the composite map

$$E \to F \to \mathbb{Z}_k.$$

This completes the proof. □

Kang and Lee proved the following in their paper [**?**].

THEOREM 2.3. *Let M be a 3-dimensional Seifert manifold modeled on $\widetilde{PSL(2, \mathbb{R})}$. Any isometry of M can be isotoped to a fixed point free isometry.*

So we can obtain the conclusion of this article:

COROLLARY 2.4. *Let M be a closed 3-manifold modeled on $\widetilde{PSL(2,\mathbb{R})}$. If f is homotopically periodic, then $N(f) = L(f) = 0$.*

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