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ON THE HYERS–ULAM–RASSIAS STABILITY OF THE JENSEN'S EQUATION IN BANACH MODULES

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ABSTRACT. We prove the Hyers-Ulam-Rassias stability of the Jensen's equation in Banach modules over a Banach algebra.

1. Introduction

Let E_1 and E_2 be Banach spaces, and $f : E_1 \to E_2$ a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th.M. Rassias [7] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in E_1$.

The stability problems of functional equations have been investigated in several papers ([2, 3, 4, 5]).

Throughout this paper, let B be a unital Banach algebra with norm $|\cdot|$, and let ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ be left Banach B-modules with norms $||\cdot||$ and $||\cdot||$, respectively.

We are going to prove the Hyers-Ulam-Rassias stability of the Jensen's equation in Banach modules over a Banach algebra.

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THEOREM 1. *Let $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ be a mapping for which there exists a function $\varphi : {}_B\mathbb{B}_1 \setminus \{0\} \times {}_B\mathbb{B}_1 \setminus \{0\} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 3^{-k} \varphi(3^k x, 3^k y) < \infty,$$

$$\|2af(\frac{x+y}{2}) - f(ax) - f(ay)\| \leq \varphi(x, y)$$

for all $a \in B(|a| = 1)$ and all $x, y \in {}_B\mathbb{B}_1 \setminus \{0\}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-linear mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{1}{3}(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x))$$

for all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$.

Proof. By [6, Theorem 1], it follows from the inequality of the statement for $a = 1$ that there exists a unique additive mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfying the condition given in the statement. The additive mapping T given in the proof of [6, Theorem 1] is similar to the additive mapping T given in the proof of [7, Theorem]. By the same reasoning as the proof of [7, Theorem], it follows from the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$ that the additive mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is $\mathbb{R}$-linear.

By the assumption, for each $a \in B$ ($|a| = 1$),

$$\|2af(3x) - f(2ax) - f(4ax)\| \leq \varphi(2x, 4x)$$

for all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. So

$$\begin{aligned}\|af(3^n x) - f(3^n ax)\| &= \|af(3^n x) - \frac{1}{2}f(2\cdot 3^{n-1}ax) \\ &\quad - \frac{1}{2}f(4\cdot 3^{n-1}ax) + \frac{1}{2}f(2\cdot 3^{n-1}ax) \\ &\quad + \frac{1}{2}f(4\cdot 3^{n-1}ax) - f(3^n ax)\| \\ &\le \frac{1}{2}\varphi(2\cdot 3^{n-1}x, 4\cdot 3^{n-1}x) \\ &\quad + \frac{1}{2}\|2f(3^n ax) - f(2\cdot 3^{n-1}ax) - f(4\cdot 3^{n-1}ax)\| \\ &\le \frac{1}{2}\varphi(2\cdot 3^{n-1}x, 4\cdot 3^{n-1}x) + \frac{1}{2}\varphi(2\cdot 3^{n-1}ax, 4\cdot 3^{n-1}ax)\end{aligned}$$

for all $a \in B$ ($|a| = 1$) and all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. Thus $3^{-n}\|af(3^n x) - f(3^n ax)\| \to 0$ as $n \to \infty$ for all $a \in B$ ($|a| = 1$) and all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. Hence

$$T(ax) = \lim_{n\to\infty} 3^{-n} f(3^n ax) = \lim_{n\to\infty} 3^{-n} af(3^n x) = aT(x)$$

for all $a \in B$ ($|a| = 1$) and all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. Since T is $\mathbb{R}$-linear and $T(ax) = aT(x)$ for each element $a \in B$ ($|a| = 1$),

$$\begin{aligned}T(ax + by) &= T(ax) + T(by) \\ &= |a|\cdot T(\frac{a}{|a|}x) + |b|\cdot T(\frac{b}{|b|}y) \\ &= aT(x) + bT(y)\end{aligned}$$

for all $a, b \in B \setminus \{0\}$ and all $x, y \in {}_B\mathbb{B}_1$. So the unique $\mathbb{R}$-linear mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is a B-linear mapping, as desired. □

COROLLARY 1. *Let $p < 1$ and $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ a mapping such that*

$$\|2af(\frac{x+y}{2}) - f(ax) - f(ay)\| \le ||x||^p + ||y||^p$$

for all $a \in B(|a|=1)$ and all $x, y \in {}_B\mathbb{B}_1 \setminus \{0\}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-linear mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{3+3^p}{3-3^p}||x||^p$$

for all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$.

Proof. Define $\varphi : {}_B\mathbb{B}_1 \setminus \{0\} \times {}_B\mathbb{B}_1 \setminus \{0\} \to [0, \infty)$ by $\varphi(x, y) = ||x||^p + ||y||^p$ and apply Theorem 1. □

THEOREM 2. *Let B be a unital Banach $*$-algebra over $\mathbb{C}$, and B^+ the set of positive elements of B. Let $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ be a mapping for which there exists a function $\varphi : {}_B\mathbb{B}_1 \setminus \{0\} \times {}_B\mathbb{B}_1 \setminus \{0\} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 3^{-k}\varphi(3^k x, 3^k y) < \infty,$$

$$\|2af(\frac{x+y}{2}) - f(ax) - f(ay)\| \leq \varphi(x, y)$$

for all $a \in B^+(|a|=1), a = i$ and all $x, y \in {}_B\mathbb{B}_1 \setminus \{0\}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-linear mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfying the condition given in the statement of Theorem 1.

Proof. By the same reasoning as the proof of Theorem 1, there exists a unique $\mathbb{R}$-linear mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ such that the desired condition. By the same method as the proof of Theorem 1, one can show that

$$T(ax) = \lim_{n\to\infty} 3^{-n}f(3^n ax) = \lim_{n\to\infty} 3^{-n}af(3^n x) = aT(x)$$

for all $a \in B^+(|a|=1), a = i$ and all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. So

$$\begin{aligned} T(ax+by) &= aT(x) + bT(y), \\ T(ix) &= iT(x) \end{aligned}$$

for all $a, b \in B^+ \setminus \{0\}$ and all $x, y \in {}_B\mathbb{B}_1$. For any element $a \in B$, $a = a_1 + ia_2$, where $a_1 = \frac{a+a^*}{2}$ and $a_2 = \frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = a_1{}^+ - a_1{}^- + ia_2{}^+ - ia_2{}^-$, where $a_1{}^+$, $a_1{}^-$, $a_2{}^+$, and $a_2{}^-$ are positive elements (see [1, Lemma 38.8]). So

$$\begin{aligned} T(ax) =& T(a_1{}^+x - a_1{}^-x + ia_2{}^+x - ia_2{}^-x) \\ =& a_1{}^+T(x) - a_1{}^-T(x) + a_2{}^+T(ix) - a_2{}^-T(ix) \\ =& a_1{}^+T(x) - a_1{}^-T(x) + ia_2{}^+T(x) - ia_2{}^-T(x) \\ =& (a_1{}^+ - a_1{}^- + ia_2{}^+ - ia_2{}^-)T(x) \\ =& aT(x) \end{aligned}$$

for all $a \in B$ and all $x \in {}_B\mathbb{B}_1$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in B$ and all $x, y \in {}_B\mathbb{B}_1$.

Therefore, there exists a unique B-linear mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfying the condition given in the statement of Theorem 1. □

COROLLARY 2. *Let E_1 and E_2 be complex Banach spaces. Let $f : E_1 \to E_2$ be a mapping for which there exists a function $\varphi : E_1 \setminus \{0\} \times E_1 \setminus \{0\} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 3^{-k} \varphi(3^k x, 3^k y) < \infty,$$

$$\|2\lambda f(\frac{x+y}{2}) - f(\lambda x) - f(\lambda y)\| \le \varphi(x, y)$$

for $\lambda = 1, i$ and all $x, y \in E_1 \setminus \{0\}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then there exists a unique $\mathbb{C}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{1}{3}(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x))$$

for all $x \in E_1 \setminus \{0\}$.

Proof. Since $\mathbb{C}$ is a Banach algebra, the Banach spaces E_1 and E_2 are considered as Banach modules over $\mathbb{C}$. By Theorem 2, there exists a unique $\mathbb{C}$-linear mapping $T : E_1 \to E_2$ satisfying the condition given in the statement. □

Remark 1. In Corollary 1 , when $a \in B(|a| = 1)$ are replaced by $a \in B^+(|a| = 1), a = i$, the results do also hold.

THEOREM 3. *Let* $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ *be a mapping for which there exists a function* $\varphi : {}_B\mathbb{B}_1 \setminus \{0\} \times {}_B\mathbb{B}_1 \setminus \{0\} \to [0, \infty)$ *such that*

$$\widetilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 3^k \varphi(3^{-k}x, 3^{-k}y) < \infty,$$

$$\|2af(\frac{x+y}{2}) - f(ax) - f(ay)\| \leq \varphi(x, y)$$

for all $a \in B(|a| = 1)$ *and all* $x, y \in {}_B\mathbb{B}_1 \setminus \{0\}$. *If* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in {}_B\mathbb{B}_1$, *then there exists a unique* B*-linear mapping* $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ *such that*

$$\|f(x) - f(0) - T(x)\| \leq \widetilde{\varphi}(\frac{x}{3}, \frac{-x}{3}) + \widetilde{\varphi}(\frac{-x}{3}, x)$$

for all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$.

Proof. By [6, Theorem 6], it follows from the inequality of the statement for $a = 1$ that there exists a unique additive mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfying the condition given in the statement. The additive mapping T given in the proof of [6, Theorem 6] is similar to the additive mapping T given in the proof of [7, Theorem]. By the same reasoning as the proof of [7, Theorem], it follows from the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$ that the additive mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is $\mathbb{R}$-linear.

By the assumption, for each $a \in B$ ($|a| = 1$),

$$\|2af(3^{-1}x) - f(2 \cdot 3^{-2}ax) - f(4 \cdot 3^{-2}ax)\| \le \varphi(2 \cdot 3^{-2}x, 4 \cdot 3^{-2}x)$$

for all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. So

$$\begin{aligned}\|af(3^{-n}ax) - f(3^{-n}ax)\| &= \|af(3^{-n}x) - \frac{1}{2}f(2 \cdot 3^{-n-1}ax) \\ &\quad - \frac{1}{2}f(4 \cdot 3^{-n-1}ax) + \frac{1}{2}f(2 \cdot 3^{-n-1}ax) \\ &\quad + \frac{1}{2}f(4 \cdot 3^{-n-1}ax) - f(3^{-n}ax)\| \\ &\le \frac{1}{2}\varphi(2 \cdot 3^{-n-1}x, 4 \cdot 3^{-n-1}x) \\ &\quad + \frac{1}{2}\|2f(3^{-n}ax) - f(2 \cdot 3^{-n-1}ax) - f(4 \cdot 3^{-n-1}ax)\| \\ &\le \frac{1}{2}\varphi(2 \cdot 3^{-n-1}x, 4 \cdot 3^{-n-1}x) + \frac{1}{2}\varphi(2 \cdot 3^{-n-1}ax, 4 \cdot 3^{-n-1}ax)\end{aligned}$$

for all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$ and all $a \in B$ ($|a| = 1$). Thus $3^n\|af(3^{-n}x) - f(3^{-n}ax)\| \to 0$ as $n \to \infty$ for all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$ and all $a \in B$ ($|a| = 1$). Hence

$$T(ax) = \lim_{n\to\infty} 3^n f(3^{-n}ax) = \lim_{n\to\infty} 3^n f(3^{-n}ax) = aT(x)$$

for all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$ and all $a \in B$ ($|a| = 1$). By the same reasoning as the proof of Theorem 1, the unique $\mathbb{R}$-linear mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is a B-linear mapping, as desired. □

COROLLARY 3. *Let $p > 1$ and $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ a mapping such that*

$$\|2af(\frac{x+y}{2}) - f(ax) - f(ay)\| \le ||x||^p + ||y||^p$$

for all $a \in B(|a| = 1)$ and all $x, y \in {}_B\mathbb{B}_1 \setminus \{0\}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-linear mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{3^p + 3}{3^p - 3}||x||^p$$

for all $x \in {}_B\mathbb{B}_1$.

Proof. The proof is similar to the proof of Corollary 1. □

THEOREM 4. *Let B be a unital Banach $*$-algebra over $\mathbb{C}$. Let $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ be a mapping for which there exists a mapping $\varphi : {}_B\mathbb{B}_1 \setminus \{0\} \times {}_B\mathbb{B}_1 \setminus \{0\} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x,y) = \sum_{k=0}^{\infty} 3^k \varphi(3^{-k}x, 3^{-k}y) < \infty,$$
$$\|2af(\frac{x+y}{2}) - f(ax) - f(ay)\| \le \varphi(x,y)$$

for all $a \in B^+(|a| = 1), a = i$ and all $x, y \in {}_B\mathbb{B}_1 \setminus \{0\}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-linear mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfying the condition given in the statement of Theorem 3.

Proof. The proof is similar to the proof of Theorem 2. □

COROLLARY 4. *Let E_1 and E_2 be complex Banach spaces. Let $f : E_1 \to E_2$ be a mapping for which there exists a function $\varphi : E_1 \setminus \{0\} \times E_1 \setminus \{0\} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x,y) = \sum_{k=0}^{\infty} 3^k \varphi(3^{-k}x, 3^{-k}y) < \infty,$$
$$\|2\lambda f(\frac{x+y}{2}) - f(\lambda x) - f(\lambda y)\| \le \varphi(x,y)$$

for $\lambda = 1, i$ and all $x, y \in E_1 \setminus \{0\}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then there exists a unique $\mathbb{C}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \widetilde{\varphi}(\frac{x}{3}, \frac{-x}{3}) + \widetilde{\varphi}(\frac{-x}{3}, x)$$

for all $x \in E_1 \setminus \{0\}$.

Proof. The proof is similar to the proof of Corollary 2. □

Remark 2. In Corollary 3, when $a \in B(|a| = 1)$ are replaced by $a \in B^+(|a| = 1), a = i$, the results do also hold.

Remark 3. When the second inequalities given in the statements of Theorem 1 and Theorem 3 are replaced by

$$\|2a^m f(\frac{x+y}{2}) - f(a^d x) - f(a^d y)\| \le \varphi(x, y)$$

for nonnegative integers m and d, by similar methods to the proofs of Theorem 1 and Theorem 3, one can show that there exist unique $\mathbb{R}$-linear mappings $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$, satisfying the conditions given in the statements of Theorem 1 and Theorem 3, such that

$$a^m T(x) = T(a^d x)$$

for all $a \in B(|a| = 1)$ and all $x \in {}_B\mathbb{B}_1$.

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