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A GENERALIZATION OF THE MARKOV-KAKUTANI FIXED POINT THEOREM

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ABSTRACT. The purpose of this paper is to give a new generalization of the well-known common fixed point theorem due to Markov-Kakutani in general settings.

Consider a family $\mathcal{A}$ of mappings f of some set into itself. If $f(x) = x$ for all $f \in \mathcal{A}$, we say that x is a *common fixed point* for $\mathcal{A}$. As is well-known, the following is the common fixed point theorem due to Markov-Kakutani :

Let X be a non-empty compact convex subset of a locally convex Hausdorff topological vector space and let $\mathcal{A}$ be a commuting family of affine continuous mappings of X into itself. Then there exists a common fixed point $\bar{x} \in X$ for the mappings in $\mathcal{A}$, i.e., $f(\bar{x}) = \bar{x}$ for each $f \in \mathcal{A}$.

The original proof, due to Markov [8], depends on Tychonoff's fixed point theorem, and next Kakutani [6] gave a direct proof of the above theorem, e.g., see [12]. We note here that it was conjectured that whether two commuting (non-affine) mappings of a compact convex set into itself necessarily had a fixed point. But the counterexamples were given separately by Huneke [5] and Boyce [2]. Therefore, in order to assure the existence of common fixed points for commuting family,

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we shall need extra conditions, e.g., affine or equicontinuous condition [6, 8]. We remark here that, for affine mappings, fixed points have a natural geometric significance, since we make an affine mapping linear by choosing a fixed point for a new origin. Thus a common fixed point for a family of affine mappings means a possible origin with respect to which all the mappings are linear. Till now, there have been a number of common fixed point theorems in various settings, e.g., DeMarr [3], Greenleaf [4], Kakutani [6], Namioka-Asplund [9], Ryll-Nardzewski [11], and others.

Recently, set-valued maps (or multimaps) are very important tools in nonlinear analysis and convex analysis, e.g., see Aubin [1], Kim-Yuan [7]. And most of existence results have been generalized and extended to multimaps in general spaces. Till now, we can not find any generalization of the Markov-Kakutani fixed point theorem in multimap settings.

In this paper, we shall give a new generalization of the well-known common fixed point theorem due to Markov-Kakutani to multimaps in locally convex spaces.

We first introduce the notations and definitions. Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A. Let X, Y be topological vector spaces and let $T : X \to 2^Y$ be a multimap. Recall that a multimap T is said to be *convex* [10] if $\lambda T(x_1)+(1-\lambda)T(x_2) \subseteq T(\lambda x_1+(1-\lambda)x_2)$ for each $x_1, x_2 \in X$, and every $\lambda \in [0,1]$. Note that if T is single-valued, then the convexity of T is exactly the same as the affine condition in [12]. For a subset $A \subset X$, we shall denote $T(A) := \{y \in Y \mid y \in T(x) \text{ for some } x \in A\}$.

A multimap $T : X \to 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$.

Let X be an arbitrary nonempty subset of a Hausdorff topological vector space E. Let $\mathcal{A} := \{T \mid T : X \to 2^X\}$ be a family of multimaps on X into 2^X. If $x \in T(x)$ for all $T \in \mathcal{A}$, we say that x is a *common fixed point* for $\mathcal{A}$. Recall that $\mathcal{A}$ is a *commuting* family of multimaps if $S(T(x)) = T(S(x))$ for each $x \in X$ and every pair $\{S, T\} \subset \mathcal{A}$. For a multimap $T : X \to 2^E$, denote the fixed point set $\mathcal{F}(T) := \{x \in X \mid x \in T(x)\}$.

For the other standard notations and terminologies, we shall refer to [1, 12].

First, we shall need the following Fan-Glicksberg fixed point theorem which is a far reaching generalization of the Brouwer fixed point theorem in locally convex spaces and multimap settings :

Lemma 1 [12]. *Let X be a non-empty compact convex subset of a locally convex Hausdorff topological vector space and let $T : X \to 2^X$ be an upper semicontinuous multimap such that each $T(x)$ is non-empty closed convex. Then there exists a point $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.*

We shall prove the new generalization of common fixed point theorem due to Markov-Kakutani as follows :

Theorem 1. *Let X be a non-empty compact convex subset of a locally convex Hausdorff topological vector space and let $\mathcal{A}$ be a commuting family of upper semicontinuous convex multimaps such that $T(x)$ is a non-empty closed convex subset of X for each $x \in X$ and every $T \in \mathcal{A}$. If, for each pair of distinct multimaps $\{S, T\} \subseteq \mathcal{A}$ and for each $x \in \mathcal{F}(T)$, the condition*

$$(*) \qquad \text{if} \quad y \in S(x), \quad \text{then} \quad S(T(x)) \subseteq T(y)$$

holds, then $\mathcal{A}$ has a common fixed point $\bar{x} \in X$, i.e., $\bar{x} \in T(\bar{x})$ for all $T \in \mathcal{A}$.

Proof. Since each T is upper semicontinuous, by Lemma 1, $\mathcal{F}(T)$ is non-empty closed in X for each $T \in \mathcal{A}$. Also, since each T is convex, $\mathcal{F}(T)$ is convex. In fact, for each $x_1, x_2 \in \mathcal{F}(T)$ and every $\lambda \in [0,1]$, we have $\lambda x_1 + (1-\lambda)x_2 \in \lambda T(x_1) + (1-\lambda)T(x_2) \subseteq T(\lambda x_1 + (1-\lambda)x_2)$, and hence $\mathcal{F}(T)$ is convex. We now claim that $\mathcal{F}(T) \cap \mathcal{F}(S)$ is non-empty closed convex in X for each $S, T \in \mathcal{A}$. First, we consider the multimap $S : \mathcal{F}(T) \to 2^X$, which is the restriction of S on $\mathcal{F}(T)$. Then we have $S(x) \subseteq \mathcal{F}(T)$ for each $x \in \mathcal{F}(T)$. In fact, suppose the contrary, i.e., there exists a point $y \in S(x)$ such that $y \notin T(y)$. Since $\mathcal{A}$ is a commuting family of multimaps, by the assumption $(*)$, we have

$$y \in S(x) \subseteq S(T(x)) = T(S(x)) \subseteq T(y),$$

which is a contradiction. Therefore, $S : \mathcal{F}(T) \to 2^{\mathcal{F}(T)}$ is an upper semicontinuous multimap such that each $S(x)$ is non-empty closed convex. By Lemma 1 again, there exists a point $\hat{x} \in \mathcal{F}(T)$ such that $\hat{x} \in S(\hat{x})$, which proves the assertion.

Similarly we can show that $\mathcal{F}(T) \cap \mathcal{F}(S) \cap \mathcal{F}(W)$ is non-empty closed convex in X for each triple pair $\{S, T, W\} \subset \mathcal{A}$. In fact, we consider the multimap $W : \mathcal{F}(T) \cap \mathcal{F}(S) \to 2^X$, which is the restriction of W on $\mathcal{F}(T) \cap \mathcal{F}(S)$. Then we claim that $W(x) \subset \mathcal{F}(T) \cap \mathcal{F}(S)$ for each $x \in \mathcal{F}(T) \cap \mathcal{F}(S)$. Suppose the contrary, i.e., there exists a point $y \in W(x)$ such that either $y \notin T(y)$ or $y \notin S(y)$. Without loss of generality, we may assume $y \notin T(y)$. Since $\mathcal{A}$ is a commuting family of multimaps, by the assumption $(*)$ on $\{T, W\}$, we have

$$y \in W(x) \subseteq W(T(x)) = T(W(x)) \subseteq T(y),$$

which is a contradiction. Therefore, $W : \mathcal{F}(T) \cap \mathcal{F}(S) \to 2^{\mathcal{F}(T)\cap\mathcal{F}(S)}$ is an upper semicontinuous multimap such that each $W(x)$ is non-empty closed convex; and hence, by Lemma 1 again, there exists a point $\hat{x} \in \mathcal{F}(T) \cap \mathcal{F}(S)$ such that $\hat{x} \in W(\hat{x})$, which proves the assertion.

By using the induction argument, we can obtain that the family of fixed point sets $\{\mathcal{F}(T) \mid T \in \mathcal{A}\}$ in X has the finite intersection property. Since each $\mathcal{F}(T)$ is non-empty compact, we can obtain the whole intersection property for the family $\mathcal{A}$, i.e., there exists $\bar{x} \in X$ such that

$$\bar{x} \in \bigcap_{T\in\mathcal{A}} \mathcal{F}(T) \neq \emptyset.$$

Therefore, we can obtain the common fixed point $\bar{x} \in X$ for the commuting family $\mathcal{A}$. □

Remarks (1) In Theorem 1, when $\mathcal{A}$ is a commuting family of single-valued functions, then the convexity assumption on $\mathcal{A}$ is actually the affine condition on $\mathcal{A}$, i.e., for each $f \in \mathcal{A}$,

$$\lambda f(x_1) + (1-\lambda) f(x_2) = f(\lambda x_1 + (1-\lambda)x_2)$$

for each $x_1, x_2 \in X$, and every $\lambda \in [0,1]$.

(2) In Theorem 1, when $\mathcal{A}$ is a commuting family of single-valued functions, then the assumption $(*)$ is automatically satisfied. In fact, for two single-valued functions $s, t : X \to X$, if $x = t(x)$ and $y = s(x)$, then we have $t(y) = t(s(x)) = s(t(x)) = s(x) = y$; and hence the assumption $(*)$ is satisfied, and also we know that s maps $\mathcal{F}(t)$ into itself.

Therefore, the followings is an easy consequence of Theorem 1 :

Corollary 1 [Markov-Kakutani]. *Let X be a non-empty compact convex subset of a locally convex Hausdorff topological vector* *space and let $\mathcal{A}$ be a commuting family of continuous affine mappings. Then $\mathcal{A}$ has a common fixed point $\bar{x} \in X$, i.e., $\bar{x} = f(\bar{x})$ for all $f \in \mathcal{A}$.*

Finally, we shall give the following slight extension of Theorem 1 :

Theorem 2. *Let X be a non-empty compact convex subset of a locally convex Hausdorff topological vector space and let $\mathcal{A}$ be a commuting family of upper semicontinuous multimaps such that $T(x)$ is a non-empty closed convex subset of X for each $x \in X$ and every $T \in \mathcal{A}$. Suppose that one multimap $T_0 \in \mathcal{A}$ may not be convex but every $T \in \mathcal{A} \setminus \{T_0\}$ must be convex. If the condition $(*)$ in Theorem 1 holds, then $\mathcal{A}$ has a common fixed point $\bar{x} \in X$, i.e., $\bar{x} \in T(\bar{x})$ for all $T \in \mathcal{A}$.*

Proof. Let $\mathcal{B} := \mathcal{A} \setminus \{T_0\}$. Then, by Theorem 1, the set K of common fixed points of the multimaps T in $\mathcal{B}$ is a non-empty compact convex subset of X. By repeating the same argument as the proof of Theorem 1, we can show that T_0 maps K into 2^K. By Lemma 1, the multimap T_0 has a fixed point in K, and this is the desired common fixed point for all T in $\mathcal{A}$. □

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