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STABILITY OF A JENSEN TYPE FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we solve a Jensen type functional equation and prove the stability of the Jensen type functional equation.

1. Introduction

In 1940, S. M. Ulam ([9]) posed the following question concerning the stability of homomorphisms: Given a metric group $(G, +, d)$, a number $\epsilon > 0$ and a mapping $f : G \to G$ which satisfies the inequality

$$d(f(x+y), f(x)+f(y)) < \epsilon$$

for all $x, y \in G$, does there exist an automorphism $a : G \to G$ and a constant $k > 0$, depending only on G, such that for all $x \in G$

$$d(f(x), a(x)) < k\epsilon \ ?$$

This question became a source of the stability theory in the Hyers-Ulam sense.

The case of approximately additive mappings was solved by D. H. Hyers ([1]) under the assumption that X and Y are Banach spaces. In 1978, Th. M. Rassias ([7]) generalized the result of Hyers as follows: Let $f : X \to Y$ be a mapping between Banach spaces and let $0 \leq p < 1$ be fixed. If f satisfies the inequality

$$||f(x+y) - f(x) - f(y)|| \leq \theta(||x||^p + ||y||^p)$$

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for some $\theta \geq 0$ and all $x, y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$||A(x) - f(x)|| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

In 2000, T. Trif ([8]) solved the Popoviciu functional equation

$$3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) = 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]$$

and proved the stability of the Popoviciu functional equation.

In this paper we deal with a Jensen type functional equation

$$6f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) = 3\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]. \tag{1}$$

In Section 2 in this paper we solve the Jensen type functional equation (1). In Section 3 we prove the stability of the Jensen type functional equation (1).

2. Solution of the Jensen type functional equation (1)

It is well known that if X and Y are real linear spaces, then a mapping $f : X \to Y$ is a solution of the Jensen functional equation $2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$ if and only if there exist an element $B \in Y$ and an additive mapping $A : X \to Y$ such that $f(x) = A(x) + B$ for all $x \in X$. The following theorem shows that the Jensen type functional equation (1) is equivalent to the Jensen functional equation.

THEOREM 1. *Let X and Y be real linear spaces. A mapping $f : X \to Y$ satisfies (1) for all $x, y, z \in X \setminus \{0\}$ if and only if there exist an element $B \in Y$ and an additive mapping $A : X \to Y$ such that*

$$f(x) = A(x) + B$$

for all $x \in X$.

Proof. Necessity. Set $B := f(0)$ and then define a mapping $A : X \to Y$ by $A(x) := f(x) - B$. Then $A(0) = 0$ and

$$\begin{aligned} &6A\left(\frac{x+y+z}{3}\right) + A(x) + A(y) + A(z) \\ &= 3\left[A\left(\frac{x+y}{2}\right) + A\left(\frac{y+z}{2}\right) + A\left(\frac{z+x}{2}\right)\right] \end{aligned} \tag{2}$$

for all $x, y, z \in X \setminus \{0\}$. We claim that A is additive.

Putting $z = -y$ in (2) we have

$$6A\left(\frac{x}{3}\right) + A(x) + A(y) + A(-y) = 3\left[A\left(\frac{x+y}{2}\right) + A\left(\frac{x-y}{2}\right)\right] \tag{3}$$

for all $x, y \in X \setminus \{0\}$. Replacing y by x in (3) we get

$$6A\left(\frac{x}{3}\right) = A(x) - A(-x) \tag{4}$$

for all $x \in X \setminus \{0\}$. Also, (4) is true for $x = 0$ since $A(0) = 0$. Replacing x by $3x$ in (4) we get

$$A(x) = \frac{A(3x) - A(-3x)}{6} \tag{5}$$

and hence

$$A(-x) = -A(x) \tag{6}$$

for all $x \in X$. From (5) and (6) it follows that

$$3A(x) = A(3x) \tag{7}$$

for all $x \in X$. Replacing x by $3y$ in (3) and using (6) and (7) we get

$$2A(y) = A(2y) \tag{8}$$

for all $y \in X$. Finally, putting $z = -x - y$ in (2) and using (6) we get

$$A(x) + A(y) - A(x+y) = 3\left[A\left(\frac{x+y}{2}\right) - A\left(\frac{x}{2}\right) - A\left(\frac{y}{2}\right)\right] \tag{9}$$

for all $x, y \in X$. From (8) and (9) we have

$$A(x+y) = A(x) + A(y)$$

for all $x, y \in X$.

Sufficiency. This is obvious □

3. Stability of the Jensen type functional equation (1)

Let $\mathbb{R}_+$ denote the set of nonnegative real numbers. Recall that a function $H : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is homogeneous of degree $p > 0$ if it satisfies $H(tu, tv, tw) = t^p H(u, v, w)$ for all nonnegative real numbers t, u, v and w. Throughout this section X and Y will be a real normed linear space and a real Banach space, respectively. Given a mapping $f : X \to Y$, we set

$$\begin{aligned} Df(x,y,z) = 6f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) \\ - 3\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right] \end{aligned}$$

for all $x, y, z \in X$. Note that $Df(x, y, z) = 0$ for all $x, y, z \in X \setminus \{0\}$ and $f(0) = 0$ if and only if f is additive.

THEOREM 2. *Assume that* $\delta \geq 0$ *and* $0 < p < 1$. *Let a function* H *be homogeneous of degree* p. *If the mapping* $f : X \to Y$ *satisfies*

$$||Df(x,y,z)|| \leq \delta + H(||x||,||y||,||z||) \tag{10}$$

for all $x, y, z \in X \setminus \{0\}$, *then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$||f(x) - f(0) - A(x)|| \leq \frac{\delta}{4} + \frac{1}{2(3^{1-p}-1)} H(||x||,||x||,||x||) \tag{11}$$

for all $x \in X$.

Proof. Let $g : X \to Y$ be a mapping defined by $g(x) := f(x) - f(0)$. Then $g(0) = 0$ and we have

$$||Dg(x,y,z)|| \leq \delta + H(||x||,||y||,||z||) \tag{12}$$

for all $x, y, z \in X \setminus \{0\}$. Putting $y = x$ and $z = -x$ in (12) we get

$$\left|\left|6g\left(\frac{x}{3}\right) - [g(x) - g(-x)]\right|\right| \leq \delta + H(||x||,||x||,||x||) \tag{13}$$

for all $x \in X \setminus \{0\}$. Also, (13) is true for $x = 0$ since $g(0) = 0$. Replacing x by $3x$ in (13) and dividing by 6 we get

$$\left|\left|g(x) - \frac{g(3x) - g(-3x)}{2 \cdot 3}\right|\right| \leq \frac{1}{6}(\delta + 3^p H(||x||,||x||,||x||)) \tag{14}$$

for all $x \in X$. Using (14) we have

$$\begin{aligned} &\left|\left|\frac{g(3^n x) - g(-3^n x)}{2 \cdot 3^n} - \frac{g(3^{n+1}x) - g(-3^{n+1}x)}{2 \cdot 3^{n+1}}\right|\right| \\ &\leq \frac{1}{2 \cdot 3^n}\left|\left|g(3^n x) - \frac{g(3^{n+1}x) - g(-3^{n+1}x)}{2 \cdot 3}\right|\right| \\ &\quad + \frac{1}{2 \cdot 3^n}\left|\left|g(-3^n x) - \frac{g(-3^{n+1}x) - g(3^{n+1}x)}{2 \cdot 3}\right|\right| \\ &\leq \frac{1}{6}(3^{-n}\delta + 3^{n(p-1)}3^p H(||x||,||x||,||x||)) \end{aligned} \tag{15}$$

for all $x \in X$ and all positive integers n. From (15) we have

$$\left\|\frac{g(3^m x) - g(-3^m x)}{2 \cdot 3^m} - \frac{g(3^n x) - g(-3^n x)}{2 \cdot 3^n}\right\|$$
$$\leq \frac{1}{6}\left(\frac{\delta}{3^m}\sum_{k=0}^{n-m-1} 3^{-k} + 3^{(p-1)m}3^p H(||x||,||x||,||x||)\sum_{k=0}^{n-m-1} 3^{(p-1)k}\right)$$

for all $x \in X$ and all positive integers m and n with $m < n$. This shows that $\left\{\frac{g(3^n x) - g(-3^n x)}{2 \cdot 3^n}\right\}$ is a Cauchy sequence for all $x \in X$. Consequently, we can define a mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{g(3^n x) - g(-3^n x)}{2 \cdot 3^n} \tag{16}$$

for all $x \in X$. From (14) and (15), we have

$$\begin{aligned} &\left\|\frac{g(3^n x) - g(-3^n x)}{2 \cdot 3^n} - g(x)\right\| \\ &\leq \frac{1}{6}\left(\delta \sum_{k=0}^{n-1} 3^{-k} + 3^p H(||x||,||x||,||x||)\sum_{k=0}^{n-1} 3^{(p-1)k}\right) \\ &\leq \frac{\delta}{4} + \frac{1}{2(3^{1-p}-1)} H(||x||,||x||,||x||) \end{aligned} \tag{17}$$

for all $x \in X$ and all positive integers n. Taking limit in (17) as $n \to \infty$, we get (11). By (12) and (16) we have

$$\begin{aligned} &||DA(x,y,z)|| \\ &= \lim_{n\to\infty} \frac{1}{2 \cdot 3^n}||Dg(3^n x, 3^n y, 3^n z) - Dg(-3^n x, -3^n y, -3^n z)|| \\ &\leq \lim_{n\to\infty} \frac{1}{3^n}(\delta + H(||3^n x||, ||3^n y||, ||3^n z||)) \\ &= \lim_{n\to\infty}\left(\frac{\delta}{3^n} + 3^{(p-1)n} H(||x||,||y||,||z||)\right) \\ &= 0 \end{aligned}$$

for all $x, y, z \in X \setminus \{0\}$. Since $A(0) = 0$, it follows that A is additive.

Now, let $A' : X \to Y$ be another additive mapping satisfying (11). Then we have

$$\begin{aligned} ||A(x) - A'(x)|| &= 3^{-n}||A(3^n x) - A'(3^n x)|| \\ &\leq 3^{-n}(||A(3^n x) - f(3^n x) + f(0)|| \\ &\qquad + ||f(3^n x) - f(0) - A'(3^n x)||) \\ &\leq 3^{-n}\left(\frac{\delta}{2} + \frac{1}{3^{1-p} - 1} 3^{np} H(||x||, ||x||, ||x||)\right) \\ &\leq \frac{\delta}{2 \cdot 3^n} + \frac{1}{3^{1-p} - 1} 3^{(p-1)n} H(||x||, ||x||, ||x||) \end{aligned}$$

for all $x \in X$ and all positive integers n. Since

$$\lim_{n \to \infty} \left(\frac{\delta}{2 \cdot 3^n} + \frac{1}{3^{1-p} - 1} 3^{(p-1)n} H(||x||, ||x||, ||x||)\right) = 0,$$

we can conclude that $A(x) = A'(x)$ for all $x \in X$. □

THEOREM 3. *Assume that $1 < p$. Let a function H be homogeneous of degree p. If the mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$||Df(x, y, z)|| \leq H(||x||, ||y||, ||z||) \tag{18}$$

for all $x, y, z \in X \setminus \{0\}$, then there exists a unique additive mapping $A : X \to Y$ such that

$$||f(x) - A(x)|| \leq \frac{3^{p-1}}{1 - 3^{1-p}} H(||x||, ||x||, ||x||) \tag{19}$$

for all $x \in X$.

Proof. Putting $y = x$ and $z = -x$ in (18) we get

$$\left|\left|3f\left(\frac{x}{3}\right) - \frac{f(x) - f(-x)}{2}\right|\right| \leq \frac{1}{2} H(||x||, ||x||, ||x||) \tag{20}$$

for all $x \in X \setminus \{0\}$. Also, (20) is true for $x = 0$ since $f(0) = 0$. Using (20) we have

(21)
$$\begin{aligned} &\left\|\frac{f(3^{-n}x) - f(-3^{-n}x)}{2 \cdot 3^{-n}} - \frac{f(3^{-(n+1)}x) - f(-3^{-(n+1)}x)}{2 \cdot 3^{-(n+1)}}\right\| \\ &\le \frac{1}{2} \cdot 3^n \left\|3f\left(\frac{3^{-n}x}{3}\right) - \frac{f(3^{-n}x) - f(-3^{-n}x)}{2}\right\| \\ &\quad + \frac{1}{2} \cdot 3^n \left\|3f\left(\frac{-3^{-n}x}{3}\right) - \frac{f(-3^{-n}x) - f(3^{-n}x)}{2}\right\| \\ &\le \frac{1}{2} \cdot 3^{(1-p)n} H(||x||, ||x||, ||x||) \end{aligned}$$

for all $x \in X$ and all nonnegative integers n. From (21) we have

$$\begin{aligned} &\left\|\frac{f(3^{-m}x) - f(-3^{-m}x)}{2 \cdot 3^{-m}} - \frac{f(3^{-n}x) - f(-3^{-n}x)}{2 \cdot 3^{-n}}\right\| \\ &\le \frac{1}{2} \cdot 3^{(1-p)m} H(||x||, ||x||, ||x||) \sum_{k=0}^{n-m-1} 3^{(1-p)k} \end{aligned} \tag{22}$$

for all $x \in X$ and all nonnegative integers m and n with $m < n$. This shows that $\left\{\frac{f(3^{-n}x) - f(-3^{-n}x)}{2 \cdot 3^{-n}}\right\}$ is a Cauchy sequence for all $x \in X$. Consequently, we can define a mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{f(3^{-n}x) - f(-3^{-n}x)}{2 \cdot 3^{-n}} \tag{23}$$

for all $x \in X$. By (18) and (23) we have

$$\begin{aligned} &||DA(x, y, z)|| \\ &= \lim_{n\to\infty} 2^{-1} \cdot 3^n ||Df(3^{-n}x, 3^{-n}y, 3^{-n}z) - Df(-3^{-n}x, -3^{-n}y, -3^{-n}z)|| \\ &\le \lim_{n\to\infty} 3^n H(||3^{-n}x||, ||3^{-n}y||, ||3^{-n}z||) \\ &= \lim_{n\to\infty} 3^{(1-p)n} H(||x||, ||y||, ||z||) \\ &= 0 \end{aligned}$$

for all $x, y, z \in X \setminus \{0\}$. Hence A is additive.

Putting $m = 0$ in (22) we get

$$\left\| \frac{f(x) - f(-x)}{2} - \frac{f(3^{-n}x) - f(-3^{-n}x)}{2 \cdot 3^{-n}} \right\| \leq \frac{1}{2} H(||x||, ||x||, ||x||) \sum_{k=0}^{n-1} 3^{(1-p)k} \tag{24}$$

for all $x \in X$ and all $n \in N$. Taking limit in (24) as $n \to \infty$, we get

$$\left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| \leq \frac{1}{2(1 - 3^{1-p})} H(||x||, ||x||, ||x||) \tag{25}$$

for all $x \in X$. From (20) and (25) we get

(26)
$$\begin{aligned} &\left\| 3A\left(\frac{x}{3}\right) - 3f\left(\frac{x}{3}\right) \right\| \\ &\leq \left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| + \left\| \frac{f(x) - f(-x)}{2} - 3f\left(\frac{x}{3}\right) \right\| \\ &\leq \frac{1}{2(1 - 3^{1-p})} H(||x||, ||x||, ||x||) + \frac{1}{2} H(||x||, ||x||, ||x||) \\ &\leq \frac{1}{1 - 3^{1-p}} H(||x||, ||x||, ||x||) \end{aligned}$$

for all $x \in X$. Replacing x by $3x$ in (26) and dividing by 3 we get

$$||f(x) - A(x)|| \leq \frac{3^{p-1}}{1 - 3^{1-p}} H(||x||, ||x||, ||x||)$$

for all $x \in X$. The proof of the uniqueness is similar to the proof of Theorem 2. □

Define a function $H : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $H(a, b, c) = (a^p + b^p + c^p)\theta$ where $\theta \geq 0$. Then H is homogeneous of degree $p > 0$. Thus we have the following corollaries.

COROLLARY 4. *Assume that* $\delta \geq 0$ *and* $0 < p < 1$. *If the mapping* $f : X \to Y$ *satisfies*

$$||Df(x,y,z)|| \leq \delta + \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X \setminus \{0\}$, *then there is a unique additive mapping* $A : X \to Y$ *such that*

$$||f(x) - f(0) - A(x)|| \leq \frac{\delta}{4} + \frac{3^p}{2(1 - 3^{p-1})}\theta||x||^p$$

for all $x \in X$.

COROLLARY 5. *Assume that* $1 < p$. *If the mapping* $f : X \to Y$ *satisfies* $f(0) = 0$ *and*

$$||Df(x,y,z)|| \leq \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X \setminus \{0\}$, *then there is a unique additive mapping* $A : X \to Y$ *such that*

$$||f(x) - A(x)|| \leq \frac{3^p}{1 - 3^{1-p}}\theta||x||^p$$

for all $x \in X$.

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