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REGULAR HOMOMORPHISMS IN TRANSFORMATION GROUPS

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ABSTRACT. In this paper, we introduce an extended notion of regular homomorphism of minimal sets by considering a certain subgroup of the group of automorphisms of a universal minimal transfomation group

1. Introduction

Regular minimal sets, which were first introduced by Auslander([1]), may be described as the minimal subsets of enveloping semigroups of the transformation groups. Regular minimal sets were characterized as follows.

THEOREM 1.1. *([1]) Let (X,T) be a minimal set. Then the following are pairwise equivalent.*

(1) *If I is a minimal right ideal of the enveloping semigroup $E(X)$ of (X,T), then (X,T) and (I,T) are isomorphic.*
(2) *If $(x,y) \in X \times X$, then there is an endomorphism h of (X,T) such that $h(x)$ and y are proximal.*
(3) *If (x,y) is an almost periodic point of $(X \times X,T)$, then there exists an endomorphism h of (X,T) such that $h(x)=y$.*

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A minimal set which satisfies any one of the properties listed above theorem is called *regular.*

The classification of minimal transformation groups is a crucial problem in abstract topological dynamics. This study has been strengthened and extended by consideration of homomorphisms. In [8], regular homomorphism were defined by extending the notions of regular minimal sets to homomorphisms with minimal range.

THEOREM 1.2. *([8]) Given a homomorphism $\pi : X \to Y$ with X and Y minimal, the following are equivalent.*

(1) *π is regular.*
(2) *For any two points x, x' in X with $\pi(x) = \pi(x')$, there exists an endomorphism $h : X \to X$ such that $h(x)$ and x' are proximal and $\pi h = h$.*
(3) *For any two points x, x' in X with (x, x') almost periodic and $\pi(x) = \pi(x')$, there exists an endomorphism $h : X \to X$ such that $h(x) = x'$ and $\pi h = h$.*

In this paper, we define a certain subgroups of the group of automorphisms of a universal minimal transformation group to extend the notion of the regular homomorphism.

2. Preliminaries

Throughout this paper, (X, T) will denote the transformation group with compact Hausdorff phase space X. A closed nonempty subset A of X is called a *minimal subset* if for every $x \in A$, the orbit xT is a dense subset of A. A point whose orbit closure is a minimal set is called an *almost periodic point.* If X is itself minimal, we say that it is a *minimal transformation group* or a *minimal set.*

Let (X, T) and (Y, T) be transformation groups. If π is a continuous map from X to Y with $\pi(xt) = \pi(x)t$ $(x \in X,\ t \in T)$, then π is called a *homomorphism.* If Y is minimal, π is always onto. Especially, if π is onto, π is called an *epimorphism.* Endomorphism and automorphism of minimal set are defined obviously. We denote the group of automorphisms of X by $A(X)$.

We define $E(X)$, the *enveloping semigroup* of (X,T), to be the closure of T in X^X, providing X^X with its product topology. *The minimal right ideal* I is the nonempty subset of $E(X)$ with $IE(X) \subset I$, which contains no proper nonempty subset of the same property.

THEOREM 2.1. *([4], [7]) Let* $\pi : (X,T) \to (Y,T)$ *be an epimorphism.*

(1) *There exists a unique epimorphism* $\tilde{\pi} : (E(X),T) \to (E(Y),T)$ *such that* $\theta_{\pi(x)}\tilde{\pi} = \pi\theta_x$, *where* $\theta_x : E(X) \to X$ *is defined by* $\theta_x(p) = xp$. *Moreover* $\tilde{\pi}$ *is also a semigroup homomorphism.*

(2) *If* K *is a minimal right ideal of* $E(Y)$ *and* $v \in K$ *is an idempotent, then there is an idempotent* $u \in E(X)$ *belonging to some minimal right ideal of* $E(X)$ *such that* $\tilde{\pi}(u) = v$.

The compact Hausdorff space X carries a natural uniformity $\mathcal{U}[X]$ whose indices are all the neighborhoods of the diagonal in $X \times X$.

DEFINITION 2.1. Let (X,T) be a transformation group. Two points x and y of X are called *proximal* provided that for each index $\alpha \in \mathcal{U}[X]$, there exists a $t \in T$ such that $(xt, yt) \in \alpha$. The set of all proximal pairs of points is called the *proximal relation* and is denoted by $P(X)$.

LEMMA 2.1. *([6]) Let* (X,T) *be a transformation group, and* $x, y \in X$. *Then the following statements are pairwise equivalent;*

(1) $(x,y) \in P(X)$.

(2) *There exists* $p \in E(X)$ *with* $xp = yp$.

(3) *There exists a minimal right ideal* I *in* $E(X)$ *such that* $xq = yq \quad (q \in I)$.

It is noted that x is an almost periodic point of (X,T) if and only if there exists an idempotent $u \in I$ with $xu = x$.

DEFINITION 2.2. ([10]) Let (X,T) be a transformation group. Two points x and x' are said to be regular if $h(x)$ and x' are proximal for some automorphism h of X. The set of all regular pairs in X is denoted by $R(X)$.

A minimal transformation group (M,T) is said to be *universal* if every minimal transformation group with phase group T is a homomorphic image of (M,T). For any group T, a universal minimal transformation group exists and is unique up to isomorphism ([5], [1]).

DEFINITION 2.3. ([8]) Let (X,T) and (Y,T) be minimal transformation groups. A homomorphism $\pi : X \to Y$ is said to be regular if for given $\gamma : M \to X$ and $\delta : M \to X$ with $\pi\delta = \pi\gamma$, there exists an automorphism $h : X \to X$ such that $h\delta = \gamma$ and $\pi h = \pi$.

3. Properties of $S(X,\gamma)$

Let (M,T) be a universal minimal transformation group, which will be fixed from now on, and the group of automorphisms of (M,T) is denoted by G. Given a minimal transfomation group (M,T) and homomorphism $\gamma : M \to X$, Auslander[1] defined a subgroup

$$G(X,\gamma) = \{\alpha \in G \mid \gamma\alpha = \gamma\}.$$

Two different homomorphisms from (M,T) to a minimal transformation group determines conjugate subgroups. These subgroups have been studied extensively in [1].

DEFINITION 3.1. Let (M,T) be a universal minimal transformation group and let (X,T) be a minimal transformation group. Given a homomorphism $\gamma : M \to X$, we define a subgroup of G as follows.

$$(1) \qquad S(X,\gamma) = \{\alpha \in G \mid h\gamma\alpha = \gamma \;\; for\ some\ h \in A(X)\}.$$

THEOREM 3.1. *Let (X,T) be a minimal transformation group. Then $G(X,\gamma)$ is a normal subgroup of $S(X,\gamma)$.*

Proof. Suppose that $\alpha \in G(X,\gamma)$ and $\beta \in S(X,\gamma)$. We shall prove that $\beta^{-1}\alpha\beta \in G(X,\gamma)$. By the definitions of $G(X,\gamma)$ and $S(X,\gamma)$, it follows that $\gamma\alpha = \gamma$ and $h\gamma\beta = \gamma$ for some $h \in A(X)$. Therefore,

$$(2) \qquad \gamma\beta^{-1}\alpha\beta = h\gamma\beta\beta^{-1}\alpha\beta = h\gamma\alpha\beta = h\gamma\beta = \gamma.$$

Thus we have $\beta^{-1}\alpha\beta \in G(X,\gamma)$. □

Let (X,T) and (Y,T) be two minimal transformation groups. Given a homomorphism $\pi : X \to Y$, $S(Y,\pi\gamma)$ is defined obviously, i.e.,

$$(3) \qquad S(Y,\pi\gamma) = \{\alpha \in G \mid k\pi\gamma\alpha = \pi\gamma \text{ for some } k \in A(Y)\}.$$

LEMMA 3.1. *([9]) Let (X,T) be a minimal transformation group and let $x, x' \in X$ with (x,x') almost periodic. Then $(x,x') \in R(X)$ if and only if $h(x) = x'$ for some $h \in A(X)$.*

LEMMA 3.2. *Let (X,T) be a minimal transformation group and let $\gamma : M \to X$ be given. Then $\alpha \in S(X,\gamma)$ for some $\alpha \in G$ if and only if there exist x, x' in X such that (x,x') is an almost periodic point and $(x,x') \in R(X)$.*

Proof. Let $\alpha \in S(X,\gamma)$. Then $h\gamma\alpha = \gamma$ for some $h \in A(X)$. For $m \in M$, $(m,\alpha(m))$ is an almost periodic point of $(M \times M,T)$. Put $x = \gamma(m)$ and $x' = \gamma\alpha(m)$. Then (x,x') is an almost periodic point and

$$(4) \qquad h(x') = h\gamma\alpha(m) = \gamma(m) = x.$$

Therefore, $(x,x') \in R(X)$ by Lemma 3.1.
Conversely, suppose that there exist x and x' in X such that (x,x') is almost periodic and $(x,x') \in R(X)$. By Lemma 3.1., $h(x) = x'$ for some $h \in A(X)$. Since (x,x') is an almost periodic point, there exists an almost periodic point $(m,m') \in M \times M$ such that $\gamma^*(m,m') = (x,x')$, where $\gamma^* : M \times M \to X \times X$ is defined by $\gamma^*(m,m') = (\gamma(m),\gamma(m'))$. In addition, the universal minimal set M is always regular minimal. Hence we can find an $\alpha \in G$ such that $\alpha(m') = m$. Therefore,

$$(5) \qquad h\gamma\alpha(m') = h\gamma(m) = h(x) = x' = \gamma(m').$$

This means that $h\gamma\alpha = \gamma$ and consequently, $\alpha \in S(X,\gamma)$. □

THEOREM 3.2. *Let (X,T) and (Y,T) be two minimal transformation groups and let $\gamma : M \to X$, $\pi : X \to Y$ be homomorphisms. The following statements hold.*

1. *Let (x,x') be an almost periodic point of $(X\times X,T)$ with $(x,x') \in R(X)$. Then $(\pi(x),\pi(x')) \in R(Y)$ if and only if $S(X,\gamma) \subset S(Y,\pi\gamma)$.*
2. *Let (x,x') be an almost periodic of $(X\times X,T)$ with $(\pi(x),\pi(x')) \in R(Y)$. Then $(x,x') \in R(X)$ if and only if $S(Y,\pi\gamma) \subset S(X,\gamma)$.*

Proof. (1) Let $\alpha \in S(X,\gamma)$. By Lemma 3.1. and Lemma 3.2., there exists an almost periodic point (x,x') of $(X \times X,T)$ such that $h(x) = x'$, where $x = \gamma\alpha(m)$ and $x' = \gamma(m)$, as was shown in the proof of preceding lemma. i.e., $(x,x') \in R(X)$. Since $(\pi(x),\pi(x'))$ is also almost periodic and $(\pi(x),\pi(x')) \in R(Y)$ is assumed, applying Lemma 3.1 again, we get that $k\pi(x) = \pi(x')$ for some $k \in A(Y)$, and

$$(6) \qquad k\pi\gamma\alpha(m) = k\pi(x) = \pi(x') = \pi\gamma(m).$$

This implies that $k\pi\gamma\alpha = \pi\gamma$ and thus $\alpha \in S(Y, \pi\gamma)$.
Conversely, let (x, x') be almost periodic with $(x, x') \in R(X)$. Then $\alpha \in S(X, \gamma)$ for some $\alpha \in G$ by Lemma 3.2.. Since $S(X, \gamma) \subset S(Y, \pi\gamma)$, we have $\alpha \in S(Y, \pi\gamma)$. Therefore, $(\pi(x), \pi(x')) \in R(Y)$.

(2) is proved similarly. □

4. Main Results

Now we extend the notions of regular homomorphisms as follows

DEFINITION 4.1. Let (X, T) and (Y, T) be minimal transformation groups. A homomorphism $\pi : X \to Y$ is said to be s-regular if for given $\gamma : M \to X$, $\delta : M \to X$ and an automorphism $k : Y \to Y$ with $k\pi\delta = \pi\gamma$, there exists an $h \in A(X)$ such that $h\delta = \gamma$ and $k\pi = \pi h$.

Let $\pi : X \to Y$ be a homomorphism, and let $R_\pi = \{(x, x') \in X \times X \mid \pi(x) = \pi(x')\}$. Then R_π is a closed invariant equivalence relation on X.
A homomorphism $\pi : X \to Y$ is called *proximal* if each two points belonging to the same fiber are proximal, that is, $R_\pi \subset P(X)$.

THEOREM 4.1. *Let (X, T) and (Y, T) be minimal transformation groups and let $\gamma : M \to X$. If $\pi : X \to Y$ is a proximal homomorphism, then $G(Y, \pi\gamma)$ is a normal subgroup of $S(X, \gamma)$.*

Proof. Let $\alpha \in G(Y, \pi\gamma)$ and $\beta \in S(X, \gamma)$. It suffices to show that $\beta^{-1}\alpha\beta \in G(Y, \pi\gamma)$, that is, $\pi\gamma\beta^{-1}\alpha\beta = \pi\gamma$. From the definitions of $G(Y, \pi\gamma)$ and $S(X, \gamma)$, we get that

$$(7) \qquad \pi\gamma\alpha = \pi\gamma \quad and \quad h\gamma\beta = \gamma$$

for some $h \in A(X)$. Since π is proximal and $\pi\gamma\alpha = \pi\gamma$, we obtain $\gamma\alpha = \gamma$. Consequently, it follows that

$$\pi\gamma\beta^{-1}\alpha\beta = \pi h\gamma\beta\beta^{-1}\alpha\beta = \pi h\gamma\alpha\beta = \pi h\gamma\beta = \pi\gamma. \tag{8}$$

□

LEMMA 4.1. *([3]) Let (X,T) be a minimal transformation group and let $\gamma : M \to X$ and $\delta : M \to X$ be homomorphisms. Then there exists $\alpha \in G$ such that $\delta = \gamma\alpha$.*

THEOREM 4.2. *Let (X,T) and (Y,T) be minimal transformation groups and let $\pi : X \to Y$ and $\gamma : M \to X$ be homomorphisms. Then the following are equivalent :*

(1) *π is regular.*

(2) *$G(Y,\pi\gamma) \subset S(X,\gamma)$.*

Proof. (1) $\Rightarrow$ (2) : Let $\alpha \in G(Y,\pi\gamma)$. Then $\pi\gamma\alpha = \pi\gamma$. Since π is regular, for given γ and $\delta = \gamma\alpha$ with $\pi\delta = \pi\gamma$, we can find an automorphism h such that $h\gamma\alpha = h\delta = \gamma$ and $\pi h = \pi$. This implies that $\alpha \in S(X,\gamma)$.

(2) $\Rightarrow$ (1) : Let γ and δ be given with $\pi\delta = \pi\gamma$. By Lemma 4.1, there is an $\alpha \in G$ such that $\delta = \gamma\alpha$. Therefore,

$$\pi\gamma\alpha = \pi\delta = \pi\gamma. \tag{9}$$

This shows that $\alpha \in G(Y,\pi\gamma)$, and by assumption, we have $\alpha \in S(X,\gamma)$. There exists an $h \in A(X)$ such that

$$h\gamma\alpha = h\delta = \gamma. \tag{10}$$

Next, we shall show that $\pi h = \pi$. Let $m \in M$ and $x = \gamma\alpha(m)$. Since $\pi\gamma\alpha = \pi\gamma$ and $h\gamma\alpha = \gamma$, we obtain

$$\pi(x) = \pi\gamma\alpha(m) = \pi\gamma(m) = \pi h\gamma\alpha(m) = \pi h(x). \tag{11}$$

Therefore $\pi h = \pi$, and thus π is regular. □

THEOREM 4.3. *Let* (X,T) *and* (Y,T) *be minimal transformation groups and let* $\pi : X \to Y$ *and* $\gamma : M \to X$ *be homomorphisms. Then the following statements are equivalent :*

(1) π *is s-regular.*

(2) $S(Y,\pi\gamma) \subset S(X,\gamma)$.

(3) *For any* $x, x' \in X$ *with* $(\pi(x), \pi(x')) \in R(Y)$, *there exists an* $h \in A(X)$ *such that* $h(x)$ *and* x' *are proximal and* $\pi h = k\pi$ *for some* $k \in A(Y)$.

(4) *For any* $x, x' \in X$ *with* (x, x') *almost periodic and* $(\pi(x), \pi(x')) \in R(Y)$, *there exists an* $h \in A(X)$ *such that* $h(x) = x'$ *and* $\pi h = k\pi$ *for some* $k \in A(Y)$.

Proof. (1) $\Rightarrow$ (2) : Let $\alpha \in S(Y,\pi\gamma)$. Then $k\pi\gamma\alpha = \pi\gamma$ for some $k \in A(Y)$. If we let $\delta = \gamma\alpha$ and applying (1), we can find an automorphism h such that $h\gamma\alpha = h\delta = \gamma$. Therefore $\alpha \in S(X,\gamma)$.
(2) $\Rightarrow$ (3) : Suppose that $S(Y,\pi\gamma) \subset S(X,\gamma)$. Let $x, x' \in X$ with $(\pi(x), \pi(x')) \in R(Y)$. Then there exists a $g \in A(Y)$ such that

$$g\pi(x)q = \pi(x')q \tag{12}$$

for every q in a minimal right ideal J of $E(Y)$. Let $v^2 = v \in J$. By Theorem 2.1 (2), there exists an idempotent $u \in E(X)$ such that $\tilde{\pi}(u) = v$. Since $(xu, x'u)$ is almost periodic and $(\pi(x), \pi(x')) \in R(Y)$, $(\pi(xu), \pi(x'u))$ is also almost periodic and $(\pi(xu), \pi(x'u)) \in R(Y)$. By Lemma 3.2, $\alpha \in S(Y,\pi\gamma)$ for some $\alpha \in G$. Therefore

$$k\pi\gamma\alpha(m) = \pi\gamma(m) \tag{13}$$

for some $k \in A(Y)$, where $\gamma\alpha(m) = xu$ and $\gamma(m) = x'u$. Since $S(Y,\pi\gamma) \subset S(X,\gamma)$, $\alpha \in S(X,\gamma)$. that is, $h\gamma\alpha = \gamma$ for some $h \in A(X)$. This implies that

$$h(x)u = h(xu) = h\gamma\alpha(m) = \gamma(m) = x'u \tag{14}$$

and therefore $h(x)$ and x' are proximal. From (13), we obtain

$$k\pi(xu) = \pi(x'u) = \pi h(xu) \tag{15}$$

which shows that $k\pi = \pi h$.
(3) $\Rightarrow$ (*4*) : Let (x, x') be an almost periodic point and $(\pi(x), \pi(x')) \in R(Y)$. Let h be an automorphism such that $h(x)$ and x' are proximal and $\pi h = k\pi$ for some $k \in A(Y)$. It is enough to show that $h(x) = x'$. Since $h(x)$ and x' are proximal, there is a minimal right ideal I of $E(X)$ such that $h(x)p = x'p$ for all $p \in I$. Since (x, x') is an almost periodic point, we can find an idempotent $u \in I$ such that $(x, x')u = (x, x')$. Therefore,

$$h(x) = h(xu) = h(x)u = x'u = x'. \tag{16}$$

(*4*) $\Rightarrow$ (*1*) : Let γ, δ and k be given with $k\pi\delta = \pi\gamma$. Let u be an idempotent of $E(X)$, and define $\delta(u) = x$ and $\gamma(u) = x'$. Then $(x, x') = (x, x')u$ is an almost periodic point. It follows that,

$$k\pi\delta(u) = k\pi(x) = k\pi(xu) = k\pi(x)\tilde{\pi}(u), \tag{17}$$

and,

$$\pi\gamma(u) = \pi(x') = \pi(x'u) = \pi(x')\tilde{\pi}(u). \tag{18}$$

Since $k\pi\delta = \pi\gamma$, we obtain from (17), (18),

$$k\pi(x)\tilde{\pi}(u) = \pi(x')\tilde{\pi}(u), \tag{19}$$

that is , $(\pi(x), \pi(x')) \in R(Y)$. By assumption, there exists an $h \in A(X)$ such that $h(x) = x'$ and $\pi h = k\pi$. Consequently, we have $h\delta = \gamma$ and $\pi h = k\pi$. This proves that π is s-regular. □

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