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THE SAP-PERRON INTEGRAL

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ABSTRACT. In this paper, we study the sap-Perron and ap-McShane integrals. In particular, we show that the sap-Perron integral is equivalent to the ap-McShane integral.

1. Introduction

The major and minor functions are first defined using the upper and lower derivates, and then the Perron integral is defined using the major and minor functions. Similarly, the ap-major and ap-minor functions are first defined using the upper and lower approximate derivates, and then the ap-Perron integral is defined using the ap-major and ap-minor functions.

It is well-known [4] that the Perron integral is equivalent to the Henstock integral and that the ap-Perron integral is equivalent to the ap-Henstock integral.

In this paper, we change the definitions of major and minor functions by strong derivates and strong approximate derivates rather than ordinary derivates and approximate derivates, and then define the sap-Perron integrals using such major and minor functions. We also define the ap-McShane integral, and then show that the sap-Perron integral is equivalent to the ap-McShane integral.

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2. The sap-Perron and ap-McShane integrals

A function $F : [a,b] \to R$ is said to be *approximately differentiable* at $c \in [a,b]$ if there exists a measurable set $E \subseteq [a,b]$ such that $c \in E^d$ and $F|_E$ is differentiable at c. The approximate derivative of F at c is denoted by $F'_{ap}(c)$.

For a measurable function $F : [a,b] \to R$, the *upper* and *lower approximate derivates* of F at $c \in [a,b]$ are defined by

$$\overline{AD}F(c) = \inf \Big\{ \alpha \in R : c \text{ is a point of dispersion of } \{x \in [a,b] : \tfrac{F(x)-F(c)}{x-c} \geq \alpha\} \Big\};$$

$$\underline{AD}F(c) = \sup \Big\{ \beta \in R : c \text{ is a point of dispersion of } \{x \in [a,b] : \tfrac{F(x)-F(c)}{x-c} \leq \beta\} \Big\}.$$

The measurable function $F : [a,b] \to R$ is approximately differentiable at $c \in [a,b]$ if and only if $\overline{AD}F(c)$ and $\underline{AD}F(c)$ are finite and equal [4, Corollary 16.13].

Now we define the upper and lower strong approximate derivates of a measurable function.

Definition 2.1. Let $F : [a,b] \to R$ be measurable and let $c \in [a,b]$. The *upper* and *lower strong approximate derivates* of F at c are defined by

$$\overline{SAD}F(c) = \inf_E \sup \left\{ \frac{F(x) - F(y)}{x - y} : x, y \in E, x \neq y \right\};$$

$$\underline{SAD}F(c) = \sup_E \inf \left\{ \frac{F(x) - F(y)}{x - y} : x, y \in E, x \neq y \right\},$$

where the infimum and supremum are taken over all measurable sets E containing c as a density point. The function F is *strongly approximately differentiable* at $c \in [a, b]$ if $\overline{SAD}F(c)$ and $\underline{SAD}F(c)$ are finite and equal. This common value is called the *strong approximate derivative* of F at c and is denoted by $F'_{sap}(c)$.

For a measurable function $F : [a, b] \to R$, it is easy to see that

$$\begin{matrix} \underline{SAD}F \le & & \le \overline{SAD}F \\ & \underline{AD}F \le \overline{AD}F & \\ \underline{D}F \le & & \le \overline{D}F \end{matrix} .$$

Using strong approximate derivates, it is possible to define the strong approximate major and strong approximate minor functions, and then the sap-Perron integral can be defined.

Definition 2.2. Let $f : [a, b] \to R_e$ be a function.

(1) A measurable function $U : [a, b] \to R$ is an *sap-major function* of f on $[a, b]$ if $\underline{SAD}U(x) > -\infty$ and $\underline{SAD}U(x) \ge f(x)$ for all $x \in [a, b]$.
(2) A measurable function $V : [a, b] \to R$ is an *sap-minor function* of f on $[a, b]$ if $\overline{SAD}V(x) < \infty$ and $\overline{SAD}V(x) \le f(x)$ for all $x \in [a, b]$.

Suppose that U is an sap-major function and that V is an sap-minor function of f on $[a, b]$. Since $0 \le \underline{SAD}U - \overline{SAD}V = \underline{SAD}(U - V) \le \underline{AD}(U - V)$, $U - V$ is nondecreasing on $[a, b]$ by [4, Theorem 17.3]. It follows that $V_a^b \le U_a^b$ and that $0 \le U_c^d - V_c^d \le U_a^b - V_a^b$ whenever $[c, d]$ is a subinterval of $[a, b]$.

In particular,

$$\begin{aligned} -\infty &< \sup\left\{V_a^b : V \text{ is an sap-minor function of } f \text{ on } [a,b]\right\} \\ &\leq \inf\left\{U_a^b : U \text{ is an sap-major function of } f \text{ on } [a,b]\right\} \\ &< \infty. \end{aligned}$$

Definition 2.3. A function $f : [a,b] \to R_e$ is *sap-Perron integrable* on $[a,b]$ if f has at least one sap-major function and one sap-minor function on $[a,b]$ and the numbers

$$\inf\left\{U_a^b : U \text{ is an sap-major function of } f \text{ on } [a,b]\right\};$$

$$\sup\left\{V_a^b : V \text{ is an sap-minor function of } f \text{ on } [a,b]\right\}$$

are equal. This common value is called the *sap-Perron integral* of f on $[a,b]$ and is denoted by $(SAP)\int_a^b f$. The function f is sap-Perron integrable on $E \subseteq [a,b]$ if $f\chi_E$ is sap-Perron integrable on $[a,b]$.

The following theorem is an immediate consequence of the definition.

Theorem 2.4. *A function $f : [a,b] \to R_e$ is sap-Perron integrable on $[a,b]$ if and only if for each $\varepsilon > 0$ there exist an sap-major function U and an sap-minor function V of f on $[a,b]$ such that $U_a^b - V_a^b < \varepsilon$.*

An *approximate neighborhood* (or *ap-nbd*) of $x \in [a,b]$ is a measurable set $S_x \subseteq [a,b]$ containing x and having density 1 at x. For every $x \in [a,b]$, choose an ap-nbd $S_x \subseteq [a,b]$ of x. Then we say that $S = \{S_x : x \in [a,b]\}$ is a *choice* on $[a,b]$.

A (free) tagged interval $(x,[c,d])$ is said to be *subordinate* to the choice S if $c, d \in S_x$. Let $\mathcal{P} = \{(x_i,[c_i,d_i]) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping (free) tagged intervals. If $(x_i,[c_i,d_i])$ is subordinate to S for each i, then we say that $\mathcal{P}$ is subordinate to S. If $\mathcal{P}$ is subordinate to S and $[a,b] = \cup_{i=1}^n [c_i,d_i]$, then we say that $\mathcal{P}$ is a (free) tagged partition of $[a,b]$ that is subordinate to S.

A function $f : [a,b] \to R$ is said to be *ap-Henstock integrable* on $[a,b]$ if there exists a real number A with the following property: for each $\varepsilon > 0$ there exists a choice S on $[a,b]$ such that $|f(\mathcal{P}) - A| < \varepsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to S.

It is well-known [4] that the ap-Henstock integral is equivalent to the ap-Perron integral.

We now present the definition of the ap-McShane integral.

Definition 2.5. A function $f : [a,b] \to R$ is *ap-McShane integrable* on $[a,b]$ if there exists a real number A with the following property: for every $\varepsilon > 0$ there exists a choice S on $[a,b]$ such that $|f(\mathcal{P}) - A| < \varepsilon$ whenever $\mathcal{P}$ is a free tagged partition of $[a,b]$ that is subordinate to S. The real number A is called the *ap-McShane integral* of f on $[a,b]$ and is denoted by $(AM)\int_a^b f$. The function f is ap-McShane integrable on a measurable set $E \subseteq [a,b]$ if $f\chi_E$ is ap-McShane integrable on $[a,b]$.

It is clear from the definition of each integral that every McShane integrable function is ap-McShane integrable and every ap-McShane integrable function is ap-Henstock integrable.

The following two theorems show that the sap-Perron integral is equivalent to the ap-McShane integral.

Theorem 2.6. *If $f : [a,b] \to R$ is sap-Perron integrable on $[a,b]$, then f is ap-McShane integrable on $[a,b]$ and the integrals are equal.*

Proof. Let $\varepsilon > 0$. By definition, there exist an sap-major function U and an sap-minor function V of f on $[a,b]$ such that

$$-\varepsilon < V_a^b - (SAP)\int_a^b f \le 0 \le U_a^b - (SAP)\int_a^b f < \varepsilon.$$

Since $\overline{SADV} \le f \le \underline{SADU}$, for each $x \in [a,b]$ there exists an ap-nbd S_x such that

$$f(x) - \varepsilon < \frac{U(d) - U(c)}{d - c} \quad \text{and} \quad \frac{V(d) - V(c)}{d - c} < f(x) + \varepsilon$$

for all $c, d \in S_x$ with $c \neq d$. Let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \le i \le n\}$ be a free tagged partition of $[a,b]$ that is subordinate to the choice $\{S_x\}$. Then for each i,

$$\begin{aligned} V(d_i) - V(c_i) - \varepsilon(d_i - c_i) &< f(x_i)(d_i - c_i) \\ &< U(d_i) - U(c_i) + \varepsilon(d_i - c_i). \end{aligned}$$

Summing for i,

$$V_a^b - \varepsilon(b-a) < f(\mathcal{P}) < U_a^b + \varepsilon(b-a).$$

Hence, we have

$$\begin{aligned} -\varepsilon(b-a+1) &< \{f(\mathcal{P}) - V_a^b\} + \left\{V_a^b - (SAP)\int_a^b f\right\} \\ &= f(\mathcal{P}) - (SAP)\int_a^b f \\ &= \{f(\mathcal{P}) - U_a^b\} + \left\{U_a^b - (SAP)\int_a^b f\right\} \\ &< \varepsilon(b-a+1). \end{aligned}$$

This shows that

$$\left| f(\mathcal{P}) - (SAP)\int_a^b f \right| < \varepsilon(b-a+1)$$

for every free tagged partition $\mathcal{P}$ of $[a,b]$ that is subordinate to $\{S_x\}$. It follows that f is ap-McShane integrable on $[a,b]$ and $(AM)\int_a^b f = (SAP)\int_a^b f$. □

Theorem 2.7. *If $f : [a,b] \to R$ is ap-McShane integrable on $[a,b]$, then f is sap-Perron integrable on $[a,b]$.*

Proof. Let $\varepsilon > 0$. By definition, there exists a choice $S = \{S_x\}$ on $[a,b]$ such that $\left|f(\mathcal{P}) - (AM)\int_a^b f\right| < \varepsilon$ whenever $\mathcal{P}$ is a free tagged partition of $[a,b]$ that is subordinate to S. Without loss of generality, we may assume that each point of S_x is a point of density of S_x. For each $x \in (a,b]$, let

$$U(x) = \sup\Big\{f(\mathcal{P}) : \mathcal{P} \text{ is a free tagged partition of } [a,x] \text{ that is subordinate to } S\Big\};$$

$$V(x) = \inf\Big\{f(\mathcal{P}) : \mathcal{P} \text{ is a free tagged partition of } [a,x] \text{ that is subordinate to } S\Big\};$$

and let $U(a) = 0 = V(a)$. By the Saks-Henstock Lemma, the functions U and V are finite-valued on $[a,b]$. We prove that U is an sap-major function of f on $[a,b]$; the proof that V is an sap-minor function of f on $[a,b]$ is quite similar.

From the proof of [4, Theorem 17.15], it follows that U is a measurable function. Fix $c \in [a,b]$. Let $[p,q]$ be any interval with $p,q \in S_c$. For each free tagged partition $\mathcal{P}$ of $[a,p]$ that is subordinate to S, we have

$$U(q) \geq f(\mathcal{P}) + f(c)(q-p)$$

and it follows that

$$U(q) \geq U(p) + f(c)(q-p);$$
$$\tfrac{U(q)-U(p)}{q-p} \geq f(c).$$

Since p and q are arbitrary points of S_c with $p < q$, we have $\inf_{p,q\in S_c} \frac{U(q)-U(p)}{q-p} \geq f(c)$ and hence $\underline{SAD}U(c) \geq f(c) > -\infty$. This shows that U is an sap-major function of f on $[a,b]$.

Since

$$|f(\mathcal{P}_1) - f(\mathcal{P}_2)| \leq \left|f(\mathcal{P}_1) - (AM)\int_a^b f\right| + \left|(AM)\int_a^b f - f(\mathcal{P}_2)\right| < 2\varepsilon$$

for any two free tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a,b]$ that are subordinate to S, it follows that $U_a^b - V_a^b \leq 2\varepsilon$. Hence the function f is sap-Perron integrable on $[a,b]$ by Theorem 2.4. □

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