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FROM STRONG CONTINUITY TO WEAK CONTINUITY

JAE - WOON KIM

ABSTRACT. In this note, we get the conditions such that strong continuity $\Rightarrow$ weak continuity plus interiority condition(wc+ic), and continuity $\Rightarrow$ wc+ic are true. And we investigate some equivalent conditions with weak continuity, some properties of weak continuity. And we show that almost compactness is preserved by weakly continuous function, and we improve some known results with respect to strong continuity.

1. Introduction

Let X be a topological space(briefly, space) and $A \subset X$ In this note, $Cl(A)$ and $Int(A)$ denote the closure and interior of A in X, respectively. Levine introduced the concept of *strong continuity*, and showed that strong continuity implies continuity, but the converse does not hold [7], and introduced the concepts of *weak continuity*, and $w^*.c.$[6] (called *feeble continuity* in [3]). It is well known that feeble continuity and weak continuity are unrelated [6, Examples 4 and 5], continuity implies weak continuity [6], and that a function between spaces is continuous iff the function is feebly continuous and weakly continuous [3, 6]. In [2], Chew and Tong defined the concepts of the *interiority condition* and *relative continuity* of a function between two spaces, and showed that weak continuity plus interiority condition (briefly, $wc + ic$) implies continuity, and the converse of this result

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does not hold [2, p.933, Example 5]. And they showed continuity implies relative continuity, whose converse does not hold [2, Theorem 3, Example 4]. We can diagram the above results as following, where $p\times \Leftrightarrow q$ means that $p \Rightarrow q$ is true and $p \Leftarrow q$ is not true.

strong continuity $\times \Leftrightarrow$ $\times \Leftrightarrow$ weak continuity $\times \Leftrightarrow \times$ feeble continuity

continuity

$wc + ic$ $\times \Leftrightarrow$ $\times \Leftrightarrow$ relative continuity

In section 2, we show that strong continuity and $wc + ic$ are not related using examples. And we get conditions such that implications strong continuity $\Rightarrow wc+ic$ and continuity $\Rightarrow wc+ic$ are true. In section 3, we investigate some equivalent conditions with weak continuity, some properties of weak continuity. And we show that almost compactness is preserved by weakly continuous function, and we improve some known results with respect to strong continuity or continuity.

2. Results

As a generalization of a continuous function, Levine [6] gave the following definition of a weakly continuous function.

DEFINITION 2.1 [6]. A function $f : X \to Y$ is *weakly continuous* at $x \in X$ if given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subset Cl(V)$. If this condition is satisfied at each $x \in X$, then f is said to be weakly continuous.

The following result is the Theorem 1 in [6], and is in [3, p.47].

THEOREM 2.2 [6]. *The function* $f : X \to Y$ *is weakly continuous iff for each open set* V *in* $Y, f^{-1}(V) \subset Int(f^{-1}(Cl(V)))$.

It is well known that continuity implies weak continuity and the converse does not hold [6].

DEFINITION 2.3. [2]. The function $f : X \to Y$ satisfies *the interiority condition* iff $Int(f^{-1}(Cl(V))) \subset f(V)$ for each open set V in Y.

Continuity does not imply interiority condition, see the Example 5 in [2, p. 933]. And the following Example shows that interiority condition does not imply continuity.

EXAMPLE 2.4. Let $X = \{a, b, c\}, Y = \{x, y\}, \sigma = \{\phi, X, \{a\}\}$, and $\tau = \{\phi, Y, \{x\}, \{y\}\}$. Define $f : X \to Y$ by $f(a) = f(b) = x, f(c) = y$. Then f satisfies the interiority condition, but not continuous.

The following is the revision of the Theorem 5 in [2].

DEFINITION 2.5 [2]. The function $f : X \to Y$ satisfies *the weak continuity plus interiority condition* (briefly, $wc + ic$) iff $f^{-1}(V) = Int(f^{-1}(Cl(V)))$ for each open set V in Y.

The following two results are from [3] or [7].

LEMMA 2.6 [3, 7]. *The function* $f : X \to Y$ *is strongly continuous iff* $f^{-1}(V)$ *is closed for each* $V \subset Y$.

LEMMA 2.7 [7]. *The function $f : X \to Y$ is strongly continuous iff $f^{-1}(V)$ is open for each $V \subset Y$.*

The following two Examples show that wc+ic and strong continuity are not related.

EXAMPLE 2.8. Let $X = Y = \{a, b, c\}, \sigma =$ discrete topology and $\tau = \{\phi, Y, \{b, c\}\}$. Define $f : (X, \sigma) \to (Y, \tau)$ by $f(a) = b, f(b) = a, f(c) = c$. Then f is strongly continuous, but does not satisfy wc+ic.

EXAMPLE 2.9. Let $X = Y = \{a, b, c\}, \sigma = \{\phi, X, \{a, c\}, \{b, c\}, \{c\}\}$ and $\tau =$ indiscrete topology. Define $f : (X, \sigma) \to (Y, \tau)$ by $f(a) = a, f(b) = f(c) = b$. Then f satisfies $wc + ic$, but is not strongly continuous.

LEMMA 2.10 [5]. *Let $f : X \to Y$ be an open function. Then $f^{-1}(Cl(V))$
$\subset Cl(f^{-1}(V))$ for each subset V of Y.*

LEMMA 2.11. *If $f : X \to Y$ is open and continuous, $f^{-1}(Cl(V)) = Cl(f^{-1}(V))$ for each subset V of Y.*

THEOREM 2.12. *If $f : X \to Y$ is open and strongly continuous, the function satisfies $wc + ic$.*

Proof. By Lemmas 2.6, 2.7 and 2.10, we have

$$\begin{aligned} Int(f^{-1}(Cl(V)) &= f^{-1}(Cl(V)) \subset Cl(f^{-1}(V)) \\ &= f^{-1}(V) \subset f^{-1}(Cl(V)) = Int(f^{-1}(Cl(V))) \end{aligned}$$

for each open set V in Y. □

The following Example shows that $f : X \to Y$ satisfies $wc+ic$, but f is neither open nor strongly continuous.

EXAMPLE 2.13. Let $X = Y = \{a,b,c\}, \sigma = \{\phi, X, \{a,c\}, \{c\}\}$ and τ = indiscrete topology. Define $f : (X,\sigma) \to (Y,\tau)$ by $f(a) = a, f(b) = f(c) = b$.

The following Example shows that open continuity does not imply $wc+ic$. Hence, to show that open continuity implies $wc+ic$, we need a certain condition.

EXAMPLE 2.14. Let $X = Y = \{a,b,c\}$, $\sigma = \{\phi, X, \{a\}\}$ and $\tau = \{\phi, Y, \{b\}\}$. Define $f : (X,\sigma) \to (Y,\tau)$ by $f(a) = b, f(b) = a, f(c) = c$. Then f is open and continuous, but does not satisfy $wc+ic$.

Recall that an open set V in a space X is *regular* if $V = Int(Cl(V))$ [4]

THEOREM 2.15. *Let $f : X \to Y$ be open and continuous. If each open set in X is regular, the function satisfies $wc+ic$.*

Proof. Let V be any open set in Y. Then $f^{-1}(V)$ is open in X. Since $f^{-1}(V)$ is regularly open, $f^{-1}(V) = Int(Cl(f^{-1}(V)))$. And by Lemma 2.11, $Cl(f^{-1}(V)) = f^{-1}(Cl(V))$. Hence we have $f^{-1}(V) = Int(f^{-1}(Cl(V)))$ for each open set V in Y. This completes the proof. □

By Lemma 2.10, and Theorem 2.15, We have the following result.

COROLLARY 2.16. *If $f : X \to Y$ is open and every open set in X is regular, then the followings are equivalent.*

(1) f is continuous,

(2) f satisfies $wc + ic$

The following example shows that a function $f : X \to Y$ is both open and continuous, and every open set in X is regular, but f is not strongly continuous.

EXAMPLE 2.17. Let $X = Y = \{a, b, c\}$, $\sigma = \{\phi, X, \{a\}, \{b, c\}\}$, $\tau = \{\phi, Y, \{b\}, \{a, c\}\}$. Define $f : (X, \sigma) \to (Y, \tau)$ by $f(a) = b, f(b) = a$ and $f(c) = c$. Then $f : X \to Y$ is both open and continuous, and every open set in X is regular. But f is not strongly continuous.

Since the following result is immediate, we omit the proof.

COROLLARY 2.18. *Let Y be a discrete space. Then for any space $X, f : X \to Y$ satisfies $wc + ic$ iff f is strongly continuous iff f is continuous.*

Let $Fr(V) = Cl(V) - Int(V)$ be the *boundary* of a subset V of X [4, p.72].

It is well-known that the following statements are equivalent [3, p.46].

(a) $f : X \to Y$ is continuous,

(b) $Fr(f^{-1}(B)) \subset f^{-1}(Fr(B))$ for each $B \subset Y$,

(c) $f(A') \subset (f(A))'$ for each $A \subset X$, where A' is the derived set of A.

LEMMA 2.19. *If $f : X \to Y$ is open, $f^{-1}(Fr(V)) \subset Fr(f^{-1}(V))$ for each open set V in Y.*

Proof. Use Lemma 2.10. $\square$

COROLLARY 2.20. *If $f : X \to Y$ is open and continuous, then $f^{-1}(Fr(V)) = Fr(f^{-1}(V))$ for each open V in Y.*

Proof. Since V is open, $Fr(V) = Cl(V) - Int(V) = Cl(V) - V$. Hence $f^{-1}(Fr(V)) = f^{-1}(Cl(V)) - f^{-1}(V) = Cl(f^{-1}(V)) - f^{-1}(V) = Cl(f^{-1}(V))$
$-Int(f^{-1}((V))) = Fr(f^{-1}(V))$ by Lemma 2.11. $\square$

By Lemma 2.19, we have the following result.

COROLLARY 2.21. *Let $f : X \to Y$ be open. Then the following statements are equivalent.*

(*a*) *f is continuous,*
(*b*) *$Fr(f^{-1}(V)) = f^{-1}(Fr(V))$ for each open V in Y.*

Recall that $f : X \to Y$ is *feebly continuous* [3](called $w^*.c.$ in [6]) iff $f^{-1}(Fr(V))$ is closed in X for each open set V in Y. Feeble continuity and weak continuity are unrelated. See the Examples 4 and 5 in [6].

THEOREM 2.22. *Continuity implies feeble continuity.*

Proof. Let $f : X \to Y$ be continuous. Then $f^{-1}(Cl(V))$ is closed and $f^{-1}(V)$ is open in X for each open set V in Y. Hence $f^{-1}(Fr(V))$ $= f^{-1}(Cl(V)) - f^{-1}(V)$ is closed in X. $\square$

The converse of Theorem 2.22 does not hold. See the following example.

EXAMPLE 2.23. Let $X = \{a, b, c\}, Y = \{x, y\}, \sigma = \{\phi, X\}$ and $\tau = \{\phi, Y, \{x\}, \{y\}\}$. Define $f : (X, \sigma) \to (Y, \tau)$ by $f(a) = x, f(b) = f(c) = y$.

COROLLARY 2.24. *If $f : X \to Y$ satisfies $wc + ic$ or is strongly continuous, then f is feebly continuous.*

As a special case of feeble continuity, we have the following.

COROLLARY 2.25. *If $f : X \to Y$ satisfies $wc + ic$ and is strongly continuous, $f^{-1}(Fr(V)) = \phi$ for each open set V in Y.*

Proof. By Definition 2.5 and Lemma 2.7, $f^{-1}(Fr(V)) = f^{-1}(Cl(V)) - f^{-1}(V) = Int(f^{-1}(Cl(V))) - Int(f^{-1}(Cl(V))) = \phi$. □

3. Further results on weak continuity

DEFINITION 3.1 [2]. $f : X \to Y$ is *relatively continuous* iff $f^{-1}(V)$ is open in the subspace $f^{-1}(Cl(V))$ for every open set V in Y.

Continuity implies relative continuity [2, Theorem 3], but the converse does not hold [2, Example 4]

The following two examples show that relative continuity and weak continuity are not related.

EXAMPLE 3.2. Let $X = Y = \{a, b\}, \sigma = \{\phi, X, \{a\}\}$, and $\tau = \{\phi, Y, \{b\}\}$. Define $f : (X, \sigma) \to (Y, \tau)$ by $f(a) = a, f(b) = b$. Then f is weakly continuous, but not relatively continuous.

EXAMPLE 3.3. Let $X = \{a, b, c\}, Y = \{x, y, z\}, \sigma = \{\phi, X, \{a\}, \{a, b\}\}$, $\tau = \{\phi, Y, \{x\}, \{y\}, \{x, y\}, \{x, z\}\}$. Define $f : (X, \sigma) \to (Y, \tau)$ by $f(a) = x, f(b) = y, f(c) = z$. Then f is relatively continuous, but not weakly continuous.

LEMMA 3.4 [8, THEOREM 4]. *If $f : X \to Y$ is weakly continuous, $Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for each open set V in Y.*

Arya and Gupta proved that if $f : X \to Y$ is weakly continuous and $h : Y \to W$ is strongly continuous, then $h \circ f$ is strongly continuous [1, Theorem 2.3]. The following is an improvement of this.

THEOREM 3.5. *Let $f : X \to Y$ and $h : Y \to W$ be weakly continuous, then $h \circ f$ is weakly continuous.*

Proof. Let V be any open set in W. Since h is weakly continuous, $h^{-1}(V) \subset Int(h^{-1}(Cl(V)))$.

Clearly, $f^{-1}(h^{-1}(V)) \subset f^{-1}(Int(h^{-1}(Cl(V)))), Int(h^{-1}(Cl(V)))$ is open in Y. Since f is weakly continuous,

$f^{-1}(Int(h^{-1}(Cl(V)))) \subset Int(f^{-1}(Cl(Int(h^{-1}(Cl(V))))))$

$\subset Int(f^{-1}(Cl(h^{-1}(Cl(V))))) \subset Int(f^{-1}(h^{-1}(Cl(V))))$ by Lemma 3.4, hence this completes the proof. □

Let $f_\alpha : X \to Y_\alpha$ be functions, $\alpha \in A$, and $F : X \to \Pi\, Y_\alpha, F(x) = (f_\alpha(x))$, a function into a product space $\Pi\, Y_\alpha$. If F is strongly continuous, f_α is strongly continuous [1, Theorem 2.4].

COROLLARY 3.6. *If F is weakly continuous, f_α is weakly continuous, $\alpha \in A$.*

Proof. Let $p_\alpha : \Pi\ Y_\alpha \to Y_\alpha$ be the projection map. Then p_α is continuous, and hence weakly continuous. Hence $p_\alpha \circ F = f_\alpha$ is weakly continuous by Theorem 3.5. □

Recall that a space X is said to be *almost compact* [1] if every open cover of X has a finite subfamily whose closures cover X.

Arya and Gupta proved that every strongly continuous image of an almost compact space is compact [1, Theorem 5.1]. The following is an improvement of this.

THEOREM 3.7. *Let $f : X \to Y$ be onto weakly continuous, and X be almost compact. Then Y is almost compact.*

Proof. Let $\{V_\alpha\}$ be an open cover of Y. Since f is weakly continuous, $f^{-1}(V_\alpha) \subset Int(f^{-1}Cl(V_\alpha)))$, and $\{Int(f^{-1}(Cl(V_\alpha)))\}$ is an open cover of X. Since X is almost compact, $X = \bigcup(Cl(Int(f^{-1}(Cl(V_{\alpha_i})))),$ $i = 1, \cdots, n$. Hence $f(X) = Y = f(\bigcup(Cl(Int(f^{-1}(Cl(V_{\alpha_i}))))))$

$$= \bigcup f(Cl(Int(f^{-1}(Cl(V_{\alpha_i}))))) \subset \bigcup f(Cl(f^{-1}(Cl(V_{\alpha_i}))))$$

$$\subset \bigcup f(f^{-1}(Cl(V_{\alpha_i}))) \subset \bigcup Cl(V_{\alpha_i})$$ by Lemma 3.4.

This completes the proof. □

Recall that $f : X \to Y$ is *connected* if C is connected in X, $f(C)$ is connected in Y [2, p.932], and a space X is *totally disconnected* if each component of x is $\{x\}$ for each $x \in X$ [4, p.111].

It is known that if $f : X \to Y$ is weakly continuous and X is connected, then $f(X)$ is connected ([2, Theorem 1], [8, Theorem 3]). By this result, we have the following.

LMMA 3.8. *If $f : X \to Y$ is weakly continuous, f is connected.*

THEOREM 3.9. *Let X be locally connected and Y be totally disconnected. Then $f : X \to Y$ is connected iff f is weakly continuous.*

Proof. f is connected iff f is strongly continuous [1, Theorem 3.3]. Hence f is weakly continuous. The converse is Lemma 3.8. □

A function $f : X \to Y$ such that $f^{-1}(y)$ is closed for each $y \in Y$ need not necessarily be strongly continuous [1, Example 1.3].

LEMMA 3.10 [2, THEOREM 6]. *If $f : X \to Y$ is weakly continuous and Y is Hausdorff, then $f^{-1}(y)$ is closed for each $y \in Y$.*

THEOREM 3.11. *Let Y be finite Hausdorff. Then $f : X \to Y$ is weakly continuous iff f is strongly continuous.*

Proof. Let f be weakly continuous, and V be any subset of Y. Then V is finite, and $\bigcup f^{-1}(y) = f^{-1}(V), y \in V$, is closed in X by Lemma 3.10. Hence, by Lemma 2.6, f is strongly continuous. The converse is in the introduction of this note. □

We know that if X is compact, Y is Hausdorff, and $f : X \to Y$ is continuous, then f is closed. The following is an improvement of this.

THEOREM 3.12. *Let X be compact and Y be Hausdorff. If $f : X \to Y$ is weakly continuous, f is closed.*

Proof. Let F be closed in X. Then F is compact. Let $f(x) \in f(F)$. Then for each $y \notin f(F)$ there exists an open set V_x of $f(x)$ such that

$y \notin Cl(V_x)$ [4, p. 138, 1.2]. Since f is weakly continuous, there exists an open set U_x of $x \in F$ such that $f(U_x) \subset Cl(V_x)$. Since F is compact, $F \subset \bigcup U_{x_i}, i = 1, \cdots, n$. Hence $f(F) \subset f(\bigcup U_{x_i}) \subset \bigcup Cl(V_{x_i})$. Putting $Y - \bigcup Cl(V_{x_i})$, which is open set of y and disjoint from $f(F)$. This shows that $f(F)$ is closed. □

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DEPARTMENT OF MATHEMATICS EDUCATION
CHONGJU UNIVERSITY
CHONGJU 360-764, KOREA

E-mail: `kimjw@chongju.ac.kr`