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ON THE STABILITY OF THE JENSEN'S FUNCTIONAL EQUATION IN BANACH MODULES

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ABSTRACT. We prove the Hyers-Ulam-Rassias stability of the Jensen's equation in left Banach B-modules over a unital Banach algebra B.

1. Introduction

Let E_1 and E_2 be Banach spaces, and $f : E_1 \to E_2$ a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th.M. Rassias [7] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p} ||x||^p$$

for all $x \in E_1$.

The stability problems of functional equations have been investigated in several papers ([2, 3, 4, 5]).

Throughout this paper, let B be a unital Banach algebra with norm $|\cdot|$, B_1 the set of all elements of B having norm 1, and m, d are nonnegative integers, and let ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ be left Banach B-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively.

We are going to prove the Hyers-Ulam-Rassias stability of the Jensen's equation in left Banach B-modules over a unital Banach algebra B.

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THEOREM 1. *Let $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ be a mapping for which there exists a function $\varphi : {}_B\mathbb{B}_1 \setminus \{0\} \times {}_B\mathbb{B}_1 \setminus \{0\} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 3^{-k} \varphi(3^k x, 3^k y) < \infty,$$

$$\|2f(\frac{a^m x + a^m y}{2}) - a^d f(x) - a^d f(y)\| \le \varphi(x, y)$$

for all $a \in B_1$ and all $x, y \in {}_B\mathbb{B}_1 \setminus \{0\}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique $\mathbb{R}$-linear mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{1}{3}(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x))$$

for all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$ and $T(a^m x) = a^d T(x)$ for all $a \in B_1$ and all $x \in {}_B\mathbb{B}_1$.

Proof. By [6, Theorem 1], it follows from the inequality of the statement for $a = 1$ that there exists a unique additive mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfying the condition given in the statement. The additive mapping T given in the proof of [6, Theorem 1] is similar to the additive mapping T given in the proof of [7, Theorem]. By the same reasoning as the proof of [7, Theorem], it follows from the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$ that the additive mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is $\mathbb{R}$-linear.

By the assumption, for each $a \in B_1$,

$$\|2f(3^n a^m x) - a^d f(2 \cdot 3^{n-1} x) - a^d f(4 \cdot 3^{n-1} x)\| \le \varphi(2 \cdot 3^{n-1} x, 4 \cdot 3^{n-1} x)$$

for all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. Using the fact that for each $a \in B$ and each

$z \in {}_B\mathbb{B}_2$ $\|az\| \leq K|a| \cdot \|z\|$ for some $K > 0$,

$$\begin{aligned}\|f(3^n a^m x) - a^d f(3^n x)\| = &\|f(3^n a^m x) - \frac{1}{2}a^d f(2 \cdot 3^{n-1}x) \\ &- \frac{1}{2}a^d f(4 \cdot 3^{n-1}x) + \frac{1}{2}a^d f(2 \cdot 3^{n-1}x) \\ &+ \frac{1}{2}a^d f(4 \cdot 3^{n-1}x) - a^d f(3^n x)\| \\ \leq &\frac{1}{2}\varphi(2 \cdot 3^{n-1}x, 4 \cdot 3^{n-1}x) \\ &+ \frac{1}{2}K|a^d| \cdot \|2f(3^n x) - f(2 \cdot 3^{n-1}x) - f(4 \cdot 3^{n-1}x)\| \\ \leq &\frac{1+K}{2}\varphi(2 \cdot 3^{n-1}x, 4 \cdot 3^{n-1}x)\end{aligned}$$

for all $a \in B_1$ and all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. So $3^{-n}\|f(3^n a^m x) - a^d f(3^n x)\| \to 0$ as $n \to \infty$ for all $a \in B_1$ and all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. Hence

$$T(a^m x) = \lim_{n\to\infty} 3^{-n} f(3^n a^m x) = \lim_{n\to\infty} 3^{-n} a^d f(3^n x) = a^d T(x)$$

for all $a \in B_1$ and all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. So the unique $\mathbb{R}$-linear mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies the desired conditions given in the statement. □

THEOREM 2. *Let $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ be a mapping for which there exists a function $\varphi : {}_B\mathbb{B}_1 \setminus \{0\} \times {}_B\mathbb{B}_1 \setminus \{0\} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 3^k \varphi(3^{-k}x, 3^{-k}y) < \infty,$$

$$\|2f(\frac{a^m x + a^m y}{2}) - a^d f(x) - a^d f(y)\| \leq \varphi(x, y)$$

for all $a \in B_1$ and all $x, y \in {}_B\mathbb{B}_1 \setminus \{0\}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique $\mathbb{R}$-linear mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ such that

$$\|f(x) - f(0) - T(x)\| \leq \widetilde{\varphi}(\frac{x}{3}, \frac{-x}{3}) + \widetilde{\varphi}(\frac{-x}{3}, x)$$

for all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$ *and* $T(a^m x) = a^d T(x)$ *for all* $a \in B_1$ *and all* $x \in {}_B\mathbb{B}_1$.

Proof. By the same argument as the proof of Theorem 1, we have the additive mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$, which is R-linear.

By the assumption, for each $a \in B_1$,

$$\|2f(3^{-n}a^m x) - a^d f(2\cdot 3^{-n-1}x) - a^d f(4\cdot 3^{-n-1}x)\| \le \varphi(2\cdot 3^{-n-1}x, 4\cdot 3^{-n-1}x)$$

for all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. Using the fact that for each $a \in B$ and each $z \in {}_B\mathbb{B}_2$ $\|az\| \le K|a| \cdot \|z\|$ for some $K > 0$,

$$\begin{aligned}
&\|f(3^{-n}a^m x) - a^d f(3^{-n}x)\| \\
&\quad = \|f(3^{-n}a^m x) - \frac{1}{2}a^d f(2\cdot 3^{-n-1}x) - \frac{1}{2}a^d f(4\cdot 3^{-n-1}x) \\
&\quad + \frac{1}{2}a^d f(2\cdot 3^{-n-1}x) + \frac{1}{2}a^d f(4\cdot 3^{-n-1}x) - a^d f(3^{-n}x)\| \\
&\quad \le \frac{1}{2}\varphi(2\cdot 3^{-n-1}x, 4\cdot 3^{-n-1}x) \\
&\quad + \frac{1}{2}K|a^d| \cdot \|2f(3^{-n}x) - f(2\cdot 3^{-n-1}x) - f(4\cdot 3^{-n-1}x)\| \\
&\quad \le \frac{1+K}{2}\varphi(2\cdot 3^{-n-1}x, 4\cdot 3^{-n-1}x)
\end{aligned}$$

for all $a \in B_1$ and all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. So $3^n\|f(3^{-n}a^m x) - a^d f(3^{-n}x)\| \to 0$ as $n \to \infty$ for all $a \in B_1$ and all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. Hence

$$T(a^m x) = \lim_{n\to\infty} 3^n f(3^{-n}a^m x) = \lim_{n\to\infty} 3^n a^d f(3^{-n}x) = a^d T(x)$$

for all $a \in B_1$ and all $x \in {}_B\mathbb{B}_1 \setminus \{0\}$. So the unique $\mathbb{R}$-linear mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies the desired conditions given in the statement. □

Remark. When the inequalities

$$\|2f(\frac{a^m x + a^m y}{2}) - a^d f(x) - a^d f(y)\| \le \varphi(x, y)$$

in the statements of Theorem 1 and Theorem 2 are replaced by

$$\|2a^m f(\frac{x+y}{2}) - f(a^d x) - f(a^d y)\| \le \varphi(x, y)$$

for nonnegative integers m and d, by similar methods to the proofs of Theorem 1 and Theorem 2, one can show that there exist unique $\mathbb{R}$-linear mappings $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$, satisfying the conditions given in the statements of Theorem 1 and Theorem 2.

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