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SOME PROPERTIES OF THE GENERALIZED GOTTLIEB GROUPS

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ABSTRACT. We investigate the relationships between the Gottlieb groups and the generalized Gottlieb groups, and study some properties of the generalized Gottlieb groups. Lee and Woo [5] proved that $G_n(X, i_1, X\times Y) \cong G_n(X)\oplus\pi_n(Y)$. We can easily re-prove the above main theorem of [5] using some properties of the generalized Gottlieb groups, and obtain a more powerful result as follows; if $F \xrightarrow{i} E \xrightarrow{p} B$ is a homotopically trivial fibration, then $G_n(F, i, E) \cong \pi_n(B)\oplus G_n(F)$.

1. Introduction

Let $L(A, X)$ be the space of maps from A to X with the compact open topology. For a based map $f : A \to X$, $L(A, X; f)$ will denote the path component of $L(A, X)$ containing f. Let $\omega : L(A, X; f) \to X$ be the evaluation map given by $\omega(\alpha) = \alpha(*)$ for $\alpha \in L(A, X; f)$, where $*$ is a base point of A. Gottlieb [1, 2] defined and studied the evaluation subgroup $\omega_\#(\pi_n(L(X, X, 1))) = G_n(X)$ called by Gottlieb group. Woo and Kim [9] defined the generalized Gottlieb group $G_n(A, f, X)$ and showed that $\omega_\#(\pi_n(L(A, X; f), f)) = G_n(A, f, X)$. In this paper, we investigate the relationships between the Gottlieb groups and the generalized Gottlieb groups, and study some properties of the generalized Gottlieb groups $G_n(A, f, X)$. Lee and Woo [5] proved that

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$G_n(X, i_1, X \times Y) \cong G_n(X) \oplus \pi_n(Y)$ as the main theorem using completely different method. The notation in [5] differs from ours. In particular, they denoted $G_n(A, inclusion, X)$ by $G_n(X, A)$. We can easily re-prove the above main theorem of [5] using some properties of the generalized Gottlieb groups, and obtain a more powerful result as follows; if $F \xrightarrow{i} E \xrightarrow{p} B$ is a homotopically trivial fibration, then $G_n(F, i, E) \cong \pi_n(B) \oplus G_n(F)$. Throughout this paper, space means a space of the homotopy type of a locally finite connected CW complex. The base point as well as the constant map will be denoted by $*$. For simplicity, we use the same symbol for a map and its homotopy class.

2. Some properties of the generalized Gottlieb groups

Let $f : A \to X$ be a based map. A based map $\alpha : B \to X$ is called *f-cyclic* if there is a map $\alpha : B \times A \to X$ such that $\bar{\alpha} j \sim \nabla(\alpha \vee f) : B \vee A \to X$, where $j : B \vee A \to B \times A$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. We say that $\bar{\alpha}$ is an affiliated map to α. The set of all homotopy class of f-cyclic maps from B to X is denoted by $G(B; A, f, X)$. For the case $B = S^n$, $G(S^n; A, f, X)$ will be denoted by $G_n(A, f, X)$. Also, a based map $\alpha : B \to X$ is called *cyclic* if α is 1_X-cyclic. The set of all homotopy class of cyclic maps from B to X is denoted by $G(B; X)$. For the case $B = S^n$, $G(S^n, X)$ will be denoted by $G_n(X)$.

REMARK 2.1. (1) *$G_n(X) = \cap\{G_n(A, f, X) | f : A \to X$ is a map and A is a space$\}$. For $\alpha \in G_n(X)$, there is an affiliated map $\bar{\alpha} : S^n \times X \to X$. The composition*

$$A \times S^n \xrightarrow{f \times 1} X \times S^n \xrightarrow{\bar{\alpha}} X$$

establishes that $\alpha \in G_n(A, f, X)$. Since f is arbitrary, $\alpha \in \cap_f G_n(A, f, X)$. On the other hand, if we take $A = X$ and $f = 1 : X \to X$, then the converse holds.

(2) *$G_n(X, 1, X) = G_n(X)$ and $G_n(A, *, X) = \pi_n(X)$*

In general, $G_n(X) \subset G_n(A, f, X) \subset \pi_n(X)$ for any map $f : A \to X$. The following example shows that they can be different from each other.

EXAMPLE 2.2. *It is well known [2] that* $G_5(S^5) = 2\mathbb{Z}, G_n(X \times Y) \cong G_n(X) \oplus G_n(Y)$ *and* $G_2(S^2) = 0$. *Consider the inclusion* $i_1 : S^5 \to S^5 \times S^5$ *and the projection* $p_1 : S^5 \times S^5 \to S^5$. *Then we know, from the above result and Corollary 2.9, that* $G_5(S^5 \times S^5) \cong 2\mathbb{Z} \oplus 2\mathbb{Z} \neq G_5(S^5, i_1, S^5 \times S^5) \cong 2\mathbb{Z} \oplus \mathbb{Z} \neq \pi_5(S^5 \times S^5) \cong \mathbb{Z} \oplus \mathbb{Z}$. *We also know, from Proposition 2.5, that* $G_5(S^5) = G_5(S^5 \times S^5, p_1, S^5) \cong 2\mathbb{Z} \subset \pi_5(S^5)$. *On the other hand, consider the Hopf map* $\eta : S^3 \to S^2$. *Then there exists a map* $F : S^3 \times S^2 \to S^2$ *such that* $Fj \sim \nabla(\eta \vee 1)$. *Thus we know, from Corollary 2.7, that* $G_2(S^2) = 0 \subset G_2(S^3, \eta, S^2) = \pi_2(S^2)$.

Since A is a locally compact, any continuous map $\hat{\alpha} : (S^n, *) \to (L(A, X; f), f)$ corresponds to a continuous map $\alpha : S^n \times A \to X$, where $\bar{\alpha}(s, a) = \hat{\alpha}(s)(a)$. Thus we have the following proposition.

PROPOSITION 2.3. *[9] Let* $\omega : L(A, X; f) \to X$ *be the evaluation map. Then* $\omega_\#(\pi_n(L(A, X; f))) = G_n(A, f, X)$.

We can also easily show that $G_n(A, f, X)$ is a homotopy type invariant.

PROPOSITION 2.4. (1) *If* $g : X \to Y$ *is a homotopy equivalence, then* $g_\# : G_n(A, f, X) \to G_n(A, gf, Y)$ *is an isomorphism.*
(2) *If* $h : B \to A$ *is a homotopy equivalence, then* $G_n(A, f, X) = G_n(B, fh, X)$.

PROPOSITION 2.5. *If* $f : A \to X$ *has a right homotopy inverse, then* $G_n(X) = G_n(A, f, X)$.

Proof. Let $\alpha \in G_n(A, f, X)$. Then there exists a map $\alpha : S^n \times A \to X$ such that $\bar{\alpha} j \sim \nabla(\alpha \vee f)$, where $j : S^n \vee A \to S^n \times A$ is the inclusion. Let $g : X \to A$ be a right homotopy inverse of $f : A \to X$. Consider the composition $F = \bar{\alpha}(1 \times g) : S^n \times X \overset{1\times g}{\to} S^n \times A \overset{\alpha}{\to} X$.

Then $Fj' = \bar{\alpha}(1 \times g)j' = \bar{\alpha}j(1 \vee g) \sim \nabla(\alpha \vee fg) \sim \nabla(\alpha \vee 1)$, where $j' : S^n \vee X \to S^n \times X$ is the inclusion. Thus $\alpha \in G_n(X)$. □

THEOREM 2.6. *$f : A \to X$ is a cyclic map if and only if $G(B; A, f, X) = [B, X]$ for any space B.*

Proof. Let $\alpha \in [B, X]$ and $f : A \to X$ be cyclic. Then there is a map $F : A \times X \to X$ such that $Fj \sim \nabla(f \vee 1)$, where $j : A \vee X \to A \times X$ is the inclusion. Consider the map $\bar{\alpha} = F(1 \times \alpha)T : B \times A \xrightarrow{T} A \times B \xrightarrow{1\times\alpha} A \times X \xrightarrow{F} X$, where $T : B \times A \to A \times B$ is given by $T(b, a) = (a, b)$. Then α is f-cyclic and $\alpha \in G(B; A, f, X)$. On the other hand, suppose that $G(B; A, f, X) = [B, X]$ for any space B. Take $B = X$ and consider the identity map $1_X : X \to X$. Since $1_X \in G(B; A, f, X)$, $f : A \to X$ is cyclic. □

COROLLARY 2.7. *If $f : A \to X$ is cyclic, then $G_n(A, f, X) = \pi_n(X)$ for all n.*

For any fibration sequence $\cdots \to \Omega E \to \Omega B \xrightarrow{\partial} F \xrightarrow{i} E \xrightarrow{p} B$, by ([4] p.97 Proposition 11.3), $\partial : \Omega B \to F$ is cyclic. Thus we have that $G_n(\Omega B, \partial, F) = \pi_n(F)$ for all n.

THEOREM 2.8. *$G_n(A, f, X \times Y) \cong G_n(A, p_1 f, X) \oplus G_n(A, p_2 f, Y)$.*

Proof. Define $h : G_n(A, f, X \times Y) \to G_n(A, p_1 f, X) \oplus G_n(A, p_2 f, Y)$ by $h(\alpha) = (p_1\alpha, p_2\alpha)$. Since $\alpha \in G_n(A, f, X \times Y)$, there is a map $\bar{\alpha} : S^n \times A \to X \times Y$ such that $\bar{\alpha}j \sim \nabla(\alpha \vee f)$, where $j : S^n \vee A \to S^n \times A$ is the inclusion. Then consider the maps $\bar{\alpha}_1 = p_1\bar{\alpha} : S^n \times A \to X$ and $\bar{\alpha}_2 = p_2\bar{\alpha} : S^n \times A \to Y$. Then $p_1\alpha \in G_n(A, p_1 f, X)$ and $p_2\alpha \in G_n(A, p_2 f, Y)$. Thus $h(\alpha) \in G_n(A, p_1 f, X) \oplus G_n(A, p_2 f, Y)$. Clearly h is a homomorphism. Also, define $k : G_n(A, p_1 f, X) \oplus G_n(A, p_2 f, Y) \to G_n(A, f, X \times Y)$ by $k(\alpha_1, \alpha_2) = (\alpha_1 \times \alpha_2)\Delta$. Since $\alpha_1 \in G_n(A, p_1 f, X)$ and $\alpha_2 \in G_n(A, p_2 f, Y)$, there are affiliated maps $\bar{\alpha}_1 : S^n \times A \to X$ and $\bar{\alpha}_2 : S^n \times A \to Y$ respectively. Consider the map $\bar{\alpha} = (\bar{\alpha}_1 \times \bar{\alpha}_2)(1 \times T \times$

$1)(\Delta \times \Delta) : S^n \times A \overset{\Delta\times\Delta}{\to} S^n \times S^n \times A \times A \overset{1\times T\times 1}{\to} S^n \times A \times S^n \times A \overset{\bar{\alpha}_1\times\bar{\alpha}_2}{\to} X \times Y$. Since $\bar{\alpha} \sim \nabla((\alpha_1 \times \alpha_2)\Delta \vee f)$, $k(\alpha_1, \alpha_2) \in G_n(A, f, X \times Y)$. Clearly $kh = 1$ and $hk = 1$. This proves the theorem. □

In [5], Lee and Woo proved that $G_n(X, i_1, X \times Y) \cong G_n(X) \ominus \pi_n(Y)$. Their notation differs from ours. In particular, they denoted $G_n(A,$ inclusion,$X)$ by $G_n(X, A)$. From Remark 2.1(2), we can easily obtain their theorem as the following corollary.

COROLLARY 2.9. $G_n(X, i_1, X\times Y) \cong G_n(X) \oplus \pi_n(Y)$ *and* $G_n(Y, i_2, X \times Y) \cong \pi_n(X) \oplus G_n(Y)$.

A fibration $F \overset{i}{\to} E \overset{p}{\to} B$ is called *homotopically trivial* if there exist homotopy equivalences $h : E \to B \times F$ and $h_{|F} : F \to F$ such the diagram

$$\begin{array}{ccccc} F & \xrightarrow{i} & E & \xrightarrow{p} & B \\ {\scriptstyle h_{|F}}\downarrow & & {\scriptstyle h}\downarrow & & \| \\ F & \xrightarrow{i_2} & B \times F & \xrightarrow{p_1} & B \end{array}$$

is homotopy commutative. From Proposition 2.4(1), (2) and Corollary 2.9, we can obtain more powerful result.

PROPOSITION 2.10. *If* $F \overset{i}{\to} E \overset{p}{\to} B$ *is a homotopically trivial fibration, then* $G_n(F, i, E) \cong \pi_n(B) \oplus G_n(F)$.

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