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Sequential Estimation of variable width confidence interval for the mean

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ABSTRACT. Let $\{X_n, n = 1, 2, \cdots\}$ be i.i.d. random variables with the only unknown parameters mean μ and variance σ^2. We consider a sequential confidence interval CI for the mean with coverage probability $1 - \alpha$ and expected length of confidence interval E_θ(Length of CI)$/|\mu| \leq k$ (k : constant) and give some asymptotic properties of the stopping time in various limiting situations.

1. Introduction

A very useful method for constructing sequential confidence intervals of prescribed coverage probability and precision is exemplified in the treatment by Chow and Robbins(1965) of the problem of obtaining a fixed width CI for the mean of an unknown population. This is an outgrowth of ideas of Stein(1945, 1949).

Let $X_1, X_2, \cdots$ be a sequence of independent observations from some population, we want to find a confidence interval of prescribed width $2d(d > 0)$ and prescribed coverage probability $\alpha(\alpha > 0)$ for the unknown mean μ of the population in presence of unknown variance $\sigma^2(\sigma^2 > 0)$. Chow and Robbins (1965) show that as followings.

Set $v_n = \frac{1}{n}\sum_{j=1}^{n}(X_j - \overline{X}_n)^2 + \frac{1}{n}, n \geq 1$, $\overline{X}_n$ is the sample mean.

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Let $a_1, a_2, \cdots$ be any sequence of positive constants such that $\lim_{n\to\infty} a_n = a$. Define the stopping rule

$$N = \inf\{n \geq 1 \ : \ n \geq \frac{v_n \cdot {a_n}^2}{d^2}\}$$

and put confidence interval CI of μ for the given stopping rule

$$CI = [\overline{X}_N - d, \overline{X}_N + d].$$

Then under the sole assumption that $0 < \sigma^2 < \infty$

$$\lim_{d\to 0} d^2 N/a^2\sigma^2 = 1 \quad a.s.,$$

$$\lim_{d\to 0} P(\mu \in \ CI) = \alpha \quad \text{(asymptotic consistency)},$$

and

$$\lim_{d\to 0} d^2 EN/a^2\sigma^2 = 1 \quad \text{(asymptotic efficiency)}.$$

Various sequential procedures for estimating confidence interval with a fixed width have been studied. Csenski(1980), Callaert and Jassen(1981) studied the rate of convergence for $P_\theta(\mu \in I_N) \to 1-\alpha$ as $d \to 0$ and Jureckova and Visek(1984) studied in case of having the distribution function of the form $F_\varepsilon = (1-\varepsilon)F + \varepsilon\delta_x$, $\varepsilon \in (0,1)$. Sen(1981) and Sproule(1985) studied the fixed width for the U-statistic, Aert(1993) studied for M-statistic, Ghosh and Sen(1971,1972) and Huskova(1982) studied for rank statistic, and in other cases fixed width sequential confidence interval estimations were done, for examples, nonparameter(Geertsema(1970),Sen(1981)), location parameter (Chang(1992)), quantile(Gijbels and Veraverbeke(1989)), probability density function(Martinsek(1993)), and regression parameter(Gleser (1985)) and Rahbar(1995).

If d is large compared to the mean μ, then the confidence interval may not useful in practical situations in the fixed width confidence interval estimation. So in this paper, we studied the confidence interval with variable length having E_θ(Length of confidence interval)$/|\mu| \leq k$, k : constant.

Let $X_1, X_2, \cdots$ be i.i.d. random variables with the only unknown parameters mean μ and variance σ^2. Any confidence interval CI of mean μ to be considered will be subject to two requirements. The first concerns the coverage probability condition, i.e., for any given $0 < \alpha < 1$.

$$(1.1) \qquad P_\theta(\mu \in CI) \geq 1 - \alpha \quad \text{for every } \theta = (\mu, \sigma)$$

The second requirement concerns the precision of the confidence interval. In this study, we impose

$$(1.2) \qquad E_\theta(\text{Length of } CI)/|\mu| \leq k, \; k : \text{constant}$$

Throughout this paper, we define

$$\overline{X}_n = \tfrac{1}{n}\textstyle\sum_{j=1}^n X_j \qquad {S_n}^2 = \tfrac{1}{n-1}\textstyle\sum_{j=1}^n (X_j - \overline{X}_n)^2, \text{ for } n \geq 2$$

$$Z_j = \tfrac{(X_j - \mu)}{\sigma}, \; j = 1, 2, 3, \cdots, \qquad \overline{Z}_n = \tfrac{1}{n}\textstyle\sum_{j=1}^n Z_j, \text{ for } n \geq 2$$

$${S_n}'^2 = \tfrac{1}{n-1}\textstyle\sum_{j=1}^n (Z_j - \overline{Z}_n)^2.$$

2. Sequential confidence intervals with variable length

In this section we propose and examine a sequential procedure which satisfies (1.1) and (1.2) with non-constant precision function δ where $\delta : R \to R^+$. We shall make the following assumption about $\delta(x)$

(i) $\delta(x) = o(x)$ and bounded for all $x \in R^+$

(ii) δ is differentiable and $0 < \delta'(x) < M$ for all $x \in R^+$ and for some $0 < M < \infty$.

(iii) $\delta(x+y)/\delta(x) \to 1$ as $y \to 0$ uniformly in x

We propose a reasonable stopping time N as

(2.1) $\quad N = \inf\{n \geq 2n \geq c^2 s_n^2/\delta^2(\overline{X}_n)\}, \quad c > 0$

and under the stopping rule of (2.1) take our terminal decision rule

(2.2) $\quad CI = [\overline{X}_N - \rho\delta(\overline{X}_N), \overline{X}_N + \rho\delta(\overline{X}_N)]$

with $\rho (0 < \rho < 1)$ to be chosen. □

LEMMA 2.1. $N/N_0 \xrightarrow{p} 1$ as $c \to \infty$ where $N_0 = c^2\sigma^2/\delta^2(\mu)$.

proof. Applying Chow-Robbins (1965, Lemma 1), we will get $N \to \infty$ a.s. as $c \to \infty$. By definition of N, the following double inequality on N/N_0 holds.

$$\delta^2(\mu){S_N'}^2/\delta^2(\mu + \sigma\overline{Z}_N) \leq N/N_0$$
$$< \delta^2(\mu){S_N'}^2\delta^2(\mu + \sigma\overline{Z}_{N-1}) + \delta^2(\mu)/c^2\sigma^2.$$

Since $\sigma\overline{Z}_N \to 0, \sigma\overline{Z}_{N-1} \to 0$ and $S_N' \to 1$ $a.s.$ as $c \to \infty$, the lemma holds by the assumption (iii). □

LEMMA 2.2. If δ satisfies assumption, then there exists $c_0 > 0$ such that with the stopping time (2.1) and confidence interval (2.2) for the mean μ

$$P(\mu \in CI) \geq 1 - \alpha \quad \text{for all c} \geq c_0.$$

proof.

$$P(\mu \in CI) = P(\mu \in [\overline{X}_N - \rho\delta(\overline{X}_N), \overline{X}_n + \rho\delta(\overline{X}_N)])$$
$$= P(|\overline{Z}_N| < \rho\delta(\mu + \sigma\overline{Z}_N)/\sigma)$$
$$< P(|\overline{Z}_N| < \rho\delta(\mu)/\sigma).$$

Since $\overline{Z}_N \to 0$ *a.s.* as $n \to \infty$, there exists a constant $a > 0$ such that $P(|\overline{Z}_N| > a) < \varepsilon$ for any $\varepsilon > 0$ no matter what the stopping time N is. So choose $a = \rho\frac{\delta(\mu)}{\sigma}$, then $P \geq 1 - \alpha$ for all $\{\theta = (\mu, \sigma) : \rho\frac{\delta(\mu)}{\sigma} \geq a\}$.

For $\{\theta : \rho\frac{\delta(\mu)}{\sigma} < a\}$

$$P(\mu \in CI) < P(|\sqrt{N}\overline{Z}_N| < \sqrt{N}\rho\delta(\mu)/\sigma).$$

Since $\sqrt{N}\ \overline{Z}_N$ is asymptotically standard normal by Anscombe's theorem and therefore stochastically bounded as uniformly in $\{\theta : \rho\frac{\delta(\mu)}{\sigma} < a\}$ and $\sqrt{N}\frac{\rho\delta(\mu)}{\sigma} = (N/N_0)^{\frac{1}{2}} N_0{}^{\frac{1}{2}} \frac{\delta(\mu)}{\sigma} \to \infty$ as $c \to \infty$.

Thus $P(|\sqrt{N}\ \overline{Z}_N| < \sqrt{N}\rho\delta(\mu)/\sigma) \to 1$ as $c \to \infty$ uniformly in $\{\theta : \rho\frac{\delta(\mu)}{\sigma} < a\}$. $\square$

THEOREM 2.3. Under the assumption of $\delta(x)$, there exists $c_0 > 0$ such that with the stopping time (2.1) and confidence interval (2.2)

$$E_\theta(\text{Width of } CI)/\mu < k, \quad k : \text{constant}.$$

proof. Width of $CI = 2\rho\delta(\overline{X}_N)$.

$$\delta(\overline{X}_N) = \frac{\delta(\mu)}{\sigma}\overline{Z}_N) = \delta(\mu) + \delta'(v_N)\sigma\overline{Z}_N, \quad |v_N - \mu| \leq |\sigma Z_N|.$$

$$E_\theta\delta(\overline{X}_N) = \delta(\mu) + E_\theta[\delta'(v_N)\sigma\overline{Z}_N].$$

Since $\delta'(v_N) < M$ by assumption (ii) and $E_\theta\overline{Z}_N = 0$

$$\frac{1}{\mu}E_\theta\ (2\rho\delta(\overline{X}_N)) \leq k, \quad \text{for some } \ k. \quad \square$$

THEOREM 2.4. For any given $c > 0$ and $\delta > 0$, as $\mu \to 0$

$$P(\mu \in CI) \to 2\Phi(\rho c) - 1$$

where Φ is the distribution of standard normal.

proof. $P(\mu \in CI) = P\{|\sqrt{N}\bar{Z_N}| < \sqrt{N}\rho\delta(\mu + \sigma\bar{Z_N})/\sigma\}$.

We can easily show that $\sqrt{N}\bar{Z_N} \to N(0,1)$ as $\mu \to 0$ for fixed $\sigma > 0$ and $C > 0$. and also $\sqrt{N}\rho\delta(\mu + \sigma\bar{Z_N})/\sigma = (N/N_0)^{\frac{1}{2}}\rho c\delta(\mu + \sigma\bar{Z_N})/\delta(\mu) \to \rho c$ *a.s.* as $\mu \to 0$ for any given $\sigma > 0$ and $\rho c > 0$.

Therefore $P(\mu \in CI) \to 2\Phi(\rho c) - 1$. □

We can derive the asymptotic distribution and the asymptotic expectations in various cases under stronger conditions on random variable X and the smoothness conditions on $\delta(x)$.

The only restriction on the family of distribution F_θ of random variable X are that they belong to the class

$$\{F_\theta : EZ^4 < B_0,\ E|Z^2 - 1|^3/w^3(\theta) < D_0 \text{ for some } B_0,\ D_0 > 0\}$$

where $w^2(\theta) = var(Z^2)$.

Kim(1995) shows the following asymptotic distribution for the stopping time defined in (2.1).

THEOREM 2.5. Let N be defined in (2.1) and set $N_0 = c^2\sigma^2/\delta^2(\mu)$. Then $(N - N_0)/N_0^{\frac{1}{2}} \xrightarrow{L} N(0,q)$ as $c \to \infty$ for any given μ and σ, where $q = EZ^4 - 1 + 4(\delta^{-1}(\mu)\delta'(\mu)\sigma)^2 - 4\delta^{-1}(\mu)\delta'(\mu)\sigma EZ^3$.

THEOREM 2.6. Let N be defined in (2.1) and N_0 be as in Theorem 2.5. Then

$E_\theta N/N_0 \to 1$ as $c \to \infty$ for any given μ and σ.

Kim(1995) also shows that the asymptotic properties of the stopping time N defined in (2.1) as $\mu \to 0$ for any fixed $c > 0$ and $\sigma > 0$ under stronger conditions on $\delta(x)$ than specified smoothness conditions on $\delta(x)$.

If the assumption (iii) is replaced by the stronger condition $(\delta(x + y) - \delta(x))/y\delta(x) \to 0$ as $x \to 0$ uniformly in $|y| < b$ for some $b > 0$, then we can easily show that the new smoothness condition implies assumption (iii) and $|\delta(x + y) - \delta(x)/y\delta(x)|$ is bounded for all x and for all $|y| \leq b$.

Under the above the new smoothness condition, we have the following theorem and the proof is omitted(see Kim(1995)).

THEOREM 2.7. Let N and N_0 be as in Theorem 2.5. Let $c > 0$ and $\sigma > 0$ be fixed and let $w_0 > 0$ be arbitrary . Then under the stronger smoothness conditions on $\delta(x)$.

(1) $(N - N_0)/N_0^{\frac{1}{2}} w \xrightarrow{L} N(0,1)$ as $\mu \to 0$ for $w = \sqrt{varZ_0{}^2} > w_0$ and for any given c and σ.

(2) $EN/N_0 \to 1$ as $\mu \to 0$ for any given c and σ.

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