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BOUNDED LINEAR FUNCTIONAL ON $L_a^1(B)$ RELATED WITH $\mathcal{B}_q$

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ABSTRACT. In this paper, weighted Bloch spaces $\mathcal{B}_q$ are considered on the open unit ball in $\mathbb{C}^n$. In this paper, we will show that every Bloch function in $\mathcal{B}_q$ induces a bounded linear functional on $L_a^1(B)$.

1. Introduction

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $dA(z)$ be the area measure on D normalized so that the area of D is 1. $L^2(D, dA)$ will denote the Banach space of Lebesgue measurable functions for D with

$$\| f \|_2 = \left[\int_D |f(z)|^2 dA(z) \right]^{\frac{1}{2}} < +\infty.$$

The Bergman space $L_a^2(D)$ is defined to be the subspace of $L^2(D, dA)$ consisting of analytic functions.

Since point evaluation at $z \in D$ is a bounded linear functional on the Hilbert space $L_a^2(D)$, the Riesz representation theorem implies that there exists a unique function K_z in $L_a^2(D)$ such that

$$f(z) = \int_D f(w)\overline{K_z(w)}dA(w)$$

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for all f in $L_a^2(D)$. Let $K(z,w)$ be the function on $D \times D$ defined by

$$K(z,w) = \overline{K_z(w)}.$$

$K(z,w)$ is called the Bergman kernel of D or the reproducing kernel of L_a^2 in D. It is easily proved tht $K(z,w) = 1/(1-z\overline{w})^2$(See [8]). Let $k_a(z) = K(z,a)/\sqrt{K(a,a)} = (1-|a|^2)(1-z\overline{a})^{-2}$. k_a is called the normalized reproducing kernel of $L_a^2(D)$.

Since $L_a^2(D)$ is a closed subspace of the Hilbert space $L^2(D,dA)$, there exists an orthogonal projection P from $L^2(D,dA)$ onto $L_a^2(D)$. P is called the Bergman projection. It is well known that

$$Pf(z) = \int_D K(z,w)f(w)dA(w)$$

for any $f \in L^2(D,dA)$ and $z \in A$. Let $C(\overline{D})$ be the algebra of complex continuous functions on $\overline{D}$, the Euclidean closure of D, and $C_0(D)$ the subalgebra of $C(\overline{D})$ consisting of functions f with $f(z) \to 0$ as $|z| \to 1^-$.

The Bloch space of D, denoted $\mathcal{B}(D)$, consists of analytic functions f on D such that $\sup\{(1-|z|^2)|f'(z)| : z \in D\} < +\infty$. The little Bloch space of D, denoted $\mathcal{B}_0(D)$, consists of analytic functions f on D such that $(1-|z|^2)|f'(z)| \to 0$ as $|z| \to 1^-$. It was proved in [10] that, for $f \in \mathcal{B}(D)$, the followings are equivalent :

(1)$f \in \mathcal{B}_0(D)$

(2)$f = P\varphi$ for some $\varphi \in C(\overline{D})$.

(3)$f = P\varphi$ for some $\varphi \in C_0(D)$.

In this paper, we will extend the above result to weighted Bloch space in $\mathbb{C}^n$. We will also show that every function in $\mathcal{B}_q$ induces a bounded linear functional F_g on L_a^1 with $\| F_g \| \leq C \| g \|_q$.

Throughout this paper, $\mathbb{C}^n$ will be the Cartesian product of n copies of $\mathbb{C}$. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in

$\mathbb{C}^n$, the inner product is defined by $< z, w >= \sum_{j=1}^n z_j \overline{w_j}$ and the norm by $\| z \|^2 =< z, z >$. Let N denote the set of natural numbers. A multi-index α is an ordered n-tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\alpha_j \in N, j = 1, 2, \cdots, n$. For a multi-index α and $z \in \mathbb{C}^n$, set

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!,$$

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}.$$

Let B be the open unit ball in the complex space $\mathbb{C}^n$. The linear space of all holomorphic functions $f : B \to \mathbb{C}$ which satisfy

$$\sup_{z \in B} (1 - \| z \|^2) \| \nabla f(z) \| < \infty$$

is said to the space $\mathcal{B}$ of Bloch functions on B. The little Bloch space $\mathcal{B}_0$ is the subspace of $\mathcal{B}$ consisting of those functions $f : B \to \mathbb{C}$ which satisfy

$$\lim_{\|z\| \to 1} (1 - \| z \|^2) \| \nabla f(z) \| = 0.$$

For each $q > 0$, the weighted Bloch space of B, denoted by $\mathcal{B}_q$, consists of holomorphic functions $f : B \to \mathbb{C}$ which satisfy

$$\sup_{z \in B} (1 - \| z \|^2)^q \| \nabla f(z) \| < \infty.$$

The corresponding little Bloch space $\mathcal{B}_{q,0}$ is defined by the functions f in $\mathcal{B}_q$ such that

$$\lim_{\|z\| \to 1} (1 - \| z \|^2)^q \| \nabla f(z) \| = 0.$$

Clearly, both $\mathcal{B}_q$ and $\mathcal{B}_{q,0}$ are increasing function spaces of $q > 0$. In particular, $\mathcal{B}_1 = \mathcal{B}$ and $\mathcal{B}_{1,0} = \mathcal{B}_0$. Let us define a norm on $\mathcal{B}_q$ as follows;

$$\| f \|_q = |f(0)| + \sup\{(1 - \| w \|^2)^q \| \nabla f(w) \| : w \in B\}.$$

Weighted Bloch functions were studied in [6].

It is well known that, for any $\alpha > -1$ and $z \in D$,

$$f(z) = (\alpha+1)\int_D \frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^{\alpha+2}} f(w)dA(w)$$

if f is an analytic function on D with $\int_D (1-|z|^2)^\alpha |f(z)|dA(z) < +\infty$, where dA is the normalized area measure on D(See [8]). Let

$$K_\alpha(z,w) = (\alpha+1)\frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^{2+\alpha}}$$

and define an operator P_α by

$$P_\alpha f(z) = \int_D K_\alpha(z,w)f(w)dA(w).$$

We see that $P_\alpha f = f$ for all analytic function f in $L^1(D,(1-|z|^2)^\alpha dA(z))$. So P_α is a projection onto analytic functions. Let V be the adjoint of P_2 under the usual integral pairing, that is,

$$Vf(z) = 3(1-|z|^2)^2 \int_D \frac{f(w)}{(1-z\overline{w})^4} dA(w).$$

In [10], it was shown that V embeds $\mathcal{B}_0(D)$ into $C_0(D)$.

Let ν be the Lebesgue measure in $\mathbb{C}^n$ normalized by $\nu(B) = 1$. The measure μ_q is the weighted Lebesgue measure:

$$d\mu_q = c_q(1-\parallel z \parallel^2)^q d\nu(z),$$

where $q > -1$ is fixed, and c_q is a normalization constant such that $\mu_q(B) = 1$.

In [6], it was shown that if $f \in L^1_{\mu_q}(B) \cap H(B), \quad q > -1$, then

$$f(z) = c_q \int_B \frac{(1-\parallel w \parallel^2)^q}{(1-<z,w>)^{n+q+1}} f(w)d\nu(w).$$

For any $q > 0$, let W_q denote the operator defined by

$$W_q f(z) = c_{q-1} \int_B \frac{f(w)}{(1- < z, w >)^{n+q}} d\nu(w), \quad z \in B.$$

Let $C_0(B)$ be the subspace of complex-valued continuous functions on B which vanish on the boundary, $C(\overline{B})$ the space of complex-valued continuous functions on the closed unit ball $\overline{B}$.

In this paper, We will prove that W_q maps $L^\infty(B)$ boundedly onto $\mathcal{B}_q$. For each $q > 0$, the operator W_q maps $C_0(B)$ onto $\mathcal{B}_q$. For each $q > 0$, W_q also maps $C_0(B)$ onto $\mathcal{B}_{q,0}$. For each $q > 0$, W_q also maps $C_0(B)$ onto $\mathcal{B}_{q,0}$. We will prove that if $f \in L^1_a(B)$ is bounded and $g \in \mathcal{B}_q$, then

$$\left| \int_B f(z)\overline{g(z)}(1- \| z \|^2)^{q-1} d\nu(z) \right| \leq C \| f \|_{L^1_a} \| g \|_q$$

for some constant $C > 0$ which is independent of f and g. This shows that, for $g \in \mathcal{B}_q$,

$$F_g(f) = \int_B f(z)\overline{g(z)}(1- \| z \|^2)^{q-1} d\nu(z), f \in H^\infty(B),$$

extends to a bounded linear functional on $L^1_a(B)$ with $\| F_q \| \leq C \| g \|_q$.

2. Bounded Operator W_q.

THEOREM 1. *For $z \in B$, c is real, $t > -1$, define*

$$I_{c,t}(z) = \int_B \frac{(1 - \| w \|^2)^t}{|1- < z, w > |^{n+1+c+t}} d\nu(w), \quad z \in B.$$

Then,

(i) $I_{c,t}(z)$ is bounded in B if $c < 0$;

(ii) $I_{0,t}(z) \sim -\log(1 - \| z \|^2)$ as $\| z \| \to 1^-$;

(iii) $I_{c,t}(z) \sim (1 - \| z \|^2)^{-c}$ as $\| z \| \to 1^-$ if $c > 0$.

Proof. See [7, Proposition 1.4.10]. □

THEOREM 2. *Suppose $q > 0$. Then f is in $\mathcal{B}_q$ if and only if $(1 - \| z \|^2)^{q-1}|f(z)|$ is bounded on B.*

Proof. See [6]. □

THEOREM 3. *For each $q > 0$, the operator W_q maps $C_0(B)$ onto $\mathcal{B}_q$.*

Proof. Let $f \in \mathcal{B}_q$ $(q > 0)$. Then $(1- \| z \|^2)^{q-1}|f(z)|$ is bounded in B, by Theorem 2. Since

$$f(z) = c_{q-1} \int_B \frac{(1- \| w \|^2)^{q-1} f(w)}{(1- < z, w >)^{n+q}} d\nu(w),$$

$f(z) = W_q h(z)$ where $h(w) = (1- \| w \|^2)^{q-1} f(w)$ is in $C_0(B)$. Therefore, W_q maps $C_0(B)$ onto $\mathcal{B}_q$. □

THEOREM 4. *For each $q > 0$, the operator W_q maps $L^\infty(B)$ boundedly onto $\mathcal{B}_q$.*

Proof. Let $f(z) = W_q g(z)$, where $g \in L^\infty(B)$. Then

$$f(z) = c_{q-1} \int_B \frac{g(w)}{(1- < z, w >)^{n+q}} d\nu(w),$$

$$\frac{\partial}{\partial z_i} f(z) = (n+q) c_{q-1} \int_B \frac{g(w)(-\overline{w}_i)}{(1- < z, w >)^{n+q+1}} d\nu(w),$$

$$\| \nabla f(z) \| \leq \ (n+q) c_{q-1} \| g \|_\infty \int_B \frac{d\nu(w)}{|1- < z, w > |^{n+q+1}} \ .$$

By Theorem 1,

$$\| \nabla f(z) \| \leq \ (n+q) c_{q-1} \| g \|_\infty (1- \| z \|^2)^{-q}.$$

Thus,

$$(1- \| z \|^2)^q \| \nabla f(z) \| \leq \ C \| g \|_\infty \ .$$

It is also clear that $|f(0)| \leq c_{q-1} \| g \|_\infty$. Thus,

$$\begin{aligned} \| f \|_q = & \ |f(0)| + \sup\{(1- \| w \|^2)^q \| \nabla f(w) \| \ | w \in B\} \\ \leq & \ (C + c_{q-1}) \| g \|_\infty . \end{aligned}$$

Hence, W_q maps $L^\infty(B)$ boundedly into $\mathcal{B}_q$. That Q_q maps $L^\infty(B)$ onto $\mathcal{B}_q$ follows from the proof of Theorem 3. □

THEOREM 5. *For $q \geq 1$, $\mathcal{B}_{q,0}$ is the closure of the set of polynomials in the norm topology of $\mathcal{B}_{q,0}$. In particular, $\mathcal{B}_{q,0}$ is a separable Banach space by itself.*

Proof. See [6]. □

THEOREM 6. *For each $q > 0$, the operator W_q maps $C(\overline{B})$ boundedly onto $\mathcal{B}_{q,0}$.*

Proof. By the Stone-Weierstrass approximation theorem, each function in $C(\overline{B})$ can be uniformly approximated by finite linear combinations of functions of the form $z^\alpha \overline{z}^\beta$. But

$$\begin{aligned} & W_q(z^\alpha \overline{z}^\beta) \\ & = c_{q-1} \int_B \frac{w^\alpha \overline{w}^\beta}{(1- < z, w >)^{n+q}} d\nu(w) \\ & = c_{q-1} \int_B w^\alpha \overline{w}^\beta d\nu(w) + c_{q-1} \sum_{m=1}^{\infty} \sum_{i_1+i_2+\cdots+i_n=m} \\ & \qquad \frac{(n+q+m-1)!}{m!(n+q-1)!} \frac{m!}{i_1! i_2! \dots i_n!} z^I \int_B w^\alpha \overline{w}^\beta \overline{w}^I d\nu(w), \\ & \qquad\qquad I = (i_1, i_2, \dots, i_n) \\ & = c_J z^J \qquad \text{for some } J \end{aligned}$$

by $[7, Prop.1.4.8, Prop.1.4.9]$.

Hence W_q maps finte linear combination of functions of the form $z^\alpha \overline{z}^\beta$ to polynomials. Since W_q maps $L^\infty(B)$ boundedly into $\mathcal{B}_q$ and

$\mathcal{B}_{q,0}$ is closed in $\mathcal{B}_q$, W_q maps $C(\overline{B})$ boundedly into $\mathcal{B}_{q,0}$. The "onto "parts follow from the proof of Theorem 3. □

3. Bounded linear functional on L_a^1 related with weighted Bloch function.

Let E be a normed linear space and M a closed linear subspace of E. If we define linear operations on $E/M = \{x + M : x \in E\}$ by $(x + M) + (y + M) = (x + y) + M$ and $a(x + M) = ax + M$, then $\| x + M \| = \inf\{\| x + m \| : m \in M\}$ is a quotient norm on E/M. If E is a Banach space, so is E/M under this quotient norm.

THEOREM 7. *For each $q > 0$, there exists a constant $C > 0$ such that*

$$C^{-1} \| f \|_q \leq \inf\{\| g \|_\infty : f = W_q g, g \in L^\infty(B)\} \leq C \| f \|_q$$

for all f in $\mathcal{B}_q$ and

$$C^{-1} \| f \|_q \leq \inf\{\| g \|_\infty : f = W_q g, g \in C_0(B)\} \leq C \| f \|_q$$

for all f in $\mathcal{B}_{q,0}$.

Proof. Let us define an equivalence relation on L^∞ such that $g_1 \sim g_2 \Leftrightarrow W_q g_1 = W_q g_2$. Then $L^\infty/\sim$ is the family of equivalence class $[g]$ of g. Let us define a linear operator $T : L^\infty/\sim \longrightarrow B_q$ such that $T([g]) = W_q g$. Then T is a bounded linear operator on $L^\infty/\sim$ onto B_q. Also T is 1-1. By the open mapping theorem, T^{-1} is continuous. Hence there exists constant C such that $\| T^{-1} W_q g \|_\infty \leq C \| W_q g \|_q$. i.e. $\| g \|_\infty \leq C \| f \|_q$. □

Let $L_a^1(B)$ be the set of holomorphic functions f on B such that $\| f \|_{L_a^1} = \int_B |f(z)| d\nu(z) < \infty$. Then the space L_a^1 is a Banach space with the above norm $\| f \|_{L_a^1}$.

THEOREM 8. *Suppose that $f \in L_a^1$ is bounded and $g \in \mathcal{B}_q$. Then*

$$|\int_B f(z)\overline{g(z)}(1- \| z \|^2)^{q-1} d\nu(z)| \leq C \| f \|_{L_a^1} \| g \|_q$$

for some constant $C > 0$ which is independent of f and g.

Proof. Writing $g = W_q\varphi$ for some $\varphi \in L^\infty(B)$ and applying Fubini's theorem, we have

$$\begin{aligned}
&\int_B f(z)\overline{g(z)}(1- \| z \|^2)^{q-1} d\nu(z) \\
&= c_{q-1} \int_B f(z)(1- \| z \|^2)^{q-1} d\nu(z) \int_B \frac{\overline{\varphi(w)}}{(1- < w, z >)^{n+q}} d\nu(w) \\
&= c_{q-1} \int_B \overline{\varphi(w)} d\nu(w) \int_B \frac{f(z)(1- \| z \|^2)^{q-1}}{(1- < w, z >)^{n+q}} d\nu(z) \\
&= \int_B f(w)\overline{\varphi(w)} d\nu(w).
\end{aligned}$$

Hence, $| \int_B f(z)\overline{g(z)}(1- \| z \|^2)^{q-1} d\nu(z)| \leq \| f \|_{L_a^1} \| \varphi \|_\infty$. Taking the infimum over φ and applying Theorem 7, We get a constant $C > 0$ such that

$$|\int_B f(z)\overline{g(z)}(1- \| z \|^2)^{q-1} d\nu(z)| \leq C \| f \|_{L_a^1} \| g \|_q .$$

□

The space of bounded analytic functions on B will be denoted by $H^\infty(B)$.

COROLLARY 9. *For $g \in \mathcal{B}_q$,*

$$F_g(f) = \int_B f(z)\overline{g(z)}(1- \| z \|^2)^{q-1} d\nu(z), f \in H^\infty(B),$$

extends to a bounded linear functional on $L_a^1(B)$ with $\| F_g \| \leq c \| g \|_q$

Proof. This follows from Theorem 8. □

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