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FREE CYCLIC ACTIONS OF THE 3–DIMENSIONAL NILMANIFOLD

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ABSTRACT. we shall deal with ten cases out of 15 distinct almost Bieberbach groups up to Seifert local invariant. In those cases we will show that if G is a finite abelian group acting freely on the standard nilmanifold, then G is cyclic, up to topological conjugacy.

1. Introduction

It is well known that all 3-dimensional infra-nilmanifolds are Seifert manifolds. A classification of the 3-dimensional Seifert manifolds with solvable fundamental group (amongst them infra-nilmanifolds) is found in Orlik's book ([6, Theorem 1, p.142]).

The general question of classifying finite group actions on a closed 3-manifold is very hard. For example, it is not known if every finite action on S^3 is conjugate to a linear action. However, the actions on a 3-dimensional torus can be understood easily([4, 5]).

Recently it is known that there are only 15 kinds of distinct closed 3-dimensional manifolds M with a Nil-geometry up to Seifert local invariant([1]). We shall study only free actions of finite abelian groups G on the 3-dimensional nilmanifold, up to topological conjugacy. By the works of Bieberbach and Waldhausen([2, 3, 10]), this classification problem is reduced to classifying all normal nilpotent subgroups of almost Bieberbach groups of finite index, up to affine conjugacy.

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In this paper we shall deal with ten cases out of 15 distinct almost Bieberbach groups up to Seifert local invariant. In those cases we will show that if G is a finite abelian group acting freely on the standard nilmanifold, then G is cyclic, up to topological conjugacy. All the necessary ideas and techniques for finding and classifying all possible finite abelian group actions on the 3-dimensional nilmanifold can be found in [9]. The problem will be reduced to a purely group-theoretic one.

Note that Nil denotes the 3-dimensional Heisenberg group; i.e. Nil consists of all 3×3 real upper triangular matrices with diagonal entries 1, which is connected, simply connected and two-step nilpotent. Hence Nil has the structure of a line bundle over $\mathbb{R}^2$. We take a left invariant metric coming from the orthonormal basis

$$\left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

for the Lie algebra of Nil. This is, what is called, the Nil-geometry and its isometry group is $\mathrm{Isom(Nil)} = \mathrm{Nil} \rtimes O(2)$([7, 8]). All isometries of Nil preserve orientation and the bundle structure.

Throughout this paper, we shall denote the Heisenberg group Nil simply by $\mathcal{H}$. All terminologies and notations are followed by those in [8].

2. Free cyclic actions of the 3-dimensional nilmanifold

We shall study free actions of finite abelian groups G on the *standard nilmanifold* $\mathcal{N}$ which yield an infra-nilmanifold homeomorphic to $\mathcal{H}/\Gamma$. First we shall deal with the seifert bundle type 3 case out of 15 distinct almost Bieberbach groups up to Seifert local invariant([1, Proposition 6.1]).

Let us denote Γ imbedded in $\mathrm{Aff}(\mathcal{H})$ by

$\Gamma = \langle t_1, t_2, t_3, \alpha \mid [t_1, t_2] = t_3^{3k},\ \alpha^3 = t_3,\ \alpha t_2 \alpha^{-1} = t_1,\ \alpha t_1 \alpha^{-1} = t_2^{-1} t_1^{-1} \rangle$, where $t_1 = (e_1, I), t_2 = (e_2, I), t_3^{3k} = (e_3, I)$, and

$$\alpha = \left(\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{9k} + \frac{1}{24} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \right).$$

Here I is the identity matrix in $\mathrm{Aut}(\mathcal{H}) = \mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R})$. Note that $[\Gamma, \Gamma] = \langle t_1 t_2^{-1}, t_2^3, t_3^{3k} \rangle$.

THEOREM 1. *There exists only one cyclic group $\mathbb{Z}_{9k}$ acting freely, up to topological conjugacy, on $\mathcal{N}$ which yields an orbit manifold homeomorphic to $\mathcal{H}/\Gamma$. The action of $\Gamma/N = \mathbb{Z}_{9k}$, where $N = \langle t_1, t_2, \alpha^{9k} \rangle$, on the nilmanifold $\mathcal{H}/N$ is given by $\langle f \rangle$:*

$$f(x, y, z) = (-x + y + \frac{1}{2},\ -x,\ z + \frac{1}{2}x^2 - xy - \frac{1}{2}x + \frac{1}{9k} + \frac{1}{24}).$$

Proof. Let N be a normal nilpotent subgroup of Γ such that $G = \Gamma/N$ is abelian. Then

$$[\Gamma, \Gamma] = \langle t_1 t_2^{-1}, t_2^3, t_3^{3k} \rangle \subset N \subset \langle t_1, t_2, t_3 \rangle.$$

Suppose N contains both $t_1 t_2^{\ell_1} t_3^{\ell_2}, t_2 t_3^r$. Then N can be represented by an ordered set of generators

$$\langle t_1 t_2^{\ell_1} t_3^{\ell_2}, t_2 t_3^r, t_3^n \rangle$$

The details can be found in [9]. By the right action of $\begin{pmatrix} 1 & 0 \\ -\ell_1 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}) \subset \mathrm{Aut}(\mathcal{H})$ on N, N reduces to $\langle t_1 t_3^{\ell},\ t_2 t_3^r,\ t_3^n \rangle$. Since N contains t_2^3 and $t_1 t_2^{-1}$, we have $t_3^{\ell - r},\ t_3^{3r} \in N$. Thus $\ell - r$ and $3r$ must be multiples of n.

Note that $0 \leq \ell, r < n$. Thus $\ell = r$ and $\ell = 0,\ \frac{n}{3},\ \frac{2n}{3}$. Since

$$[t_1t_3^{\ell},\ t_2t_3^{r}] = [t_1,\ t_2] = t_3^{3k} \in N,$$

$3k$ must be a multiple of n. Note that we shall do only an infra-nilmanifold M_1 case, i.e., $n = 3k$. Therefore the possible normal nilpotent subgroups are

$$N_1 = \langle t_1,\ t_2,\ t_3^{3k}\rangle, N_2 = \langle t_1t_3^{k},\ t_2t_3^{k},\ t_3^{3k}\rangle, N_3 = \langle t_1t_3^{2k},\ t_2t_3^{2k},\ t_3^{3k}\rangle$$

It is not hard to see $N_1 \underset{R}{\sim} N_2$ and $N_3 \underset{R}{\sim} N_4$ by using $\begin{bmatrix} 1 & \frac{1}{3} & * \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$, and $N_1 \underset{R}{\sim} N_3$ by using $\begin{bmatrix} 1 & \frac{2}{3} & * \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$. Thus we get

$$N = N_1 = \langle t_1,\ t_2,\ t_3^{3k}\rangle = \langle t_1,\ t_2,\ \alpha^{9k}\rangle.$$

Therefore there exists only one $\mathbb{Z}_{9k} = \Gamma/N$ free action on the nilmanifold $\mathcal{H}/N$ which yields an infra-nilmanifold homeomorphic to $\mathcal{H}/\Gamma$.

Suppose N does not contain either $t_1t_2^{\ell_1}t_3^{\ell_2}$ or $t_2t_3^{r}$. Since G is not abelian in these cases, we induce a contradiction.

The realization of the action of $G \cong \Gamma/N$ on the nilmanifold $\mathcal{H}/N$, as an affine action on the standard nilmanifold, is easy provided in the "Realization" procedure in [12].

Let $N = \langle t_1,\ t_2,\ \alpha^{9k}\rangle$. Since G is generated by the images of α, it is enough to calculate conjugations of α by (I, B), where $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Aut}(\mathcal{H})$.

For $\alpha = (a, A) = \left(\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{9k} + \frac{1}{24} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\right)$, we have

$$(I, B)(a, A)(I, B)^{-1} = (B(a), A) = \left(\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{9k} + \frac{1}{24} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A\right).$$

It acts on $\mathcal{H}$ by

$$\left(\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{9k}+\frac{1}{24} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A\right) \cdot \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -x+y+\frac{1}{2} & z+\frac{1}{2}x^2-xy+\frac{1}{9k}+\frac{1}{24}-\frac{x}{2} \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore if $f : \mathcal{H} \to \mathcal{H}$ is the map generated by α, then

$$f(x,y,z) = (-x+y+\frac{1}{2}, -x, z+\frac{1}{2}x^2-xy+\frac{1}{9k}+\frac{1}{24}-\frac{x}{2}). \quad \square$$

THEOREM 2. *Suppose G is a finite abelian group acting freely on the standard nilmanifold. Then G is cyclic, up to topological conjugacy, and it is one of the following. The action of $G = \Gamma_i/N_i$ on the nilmanifold $\mathcal{H}/N_i$ is given by $\langle f_i \rangle$ (i=1,...,9):*

Γ	G	Conjugacy classes of normal nilpotent subgroups
Γ_1:	$\mathbb{Z}_{16k}$	$N_1 = \langle t_1, t_2, t_3^{4k} \rangle$
Γ_2:	$\mathbb{Z}_{16k}$	$N_2 = \langle t_1, t_2, t_3^{4k} \rangle$
Γ_3:	$\mathbb{Z}_{9k}$	$N_3 = \langle t_1, t_2, t_3^{3k} \rangle$
Γ_4:	$\mathbb{Z}_{9k-3}$	$N_4 = \langle t_1, t_2, t_3^{3k-1} \rangle$
Γ_5:	$\mathbb{Z}_{9k-6}$	$N_5 = \langle t_1, t_2, t_3^{3k-2} \rangle$
Γ_6:	$\mathbb{Z}_{36k}$	$N_6 = \langle t_1, t_2, t_3^{6k} \rangle$
Γ_7:	$\mathbb{Z}_{36k}$	$N_7 = \langle t_1, t_2, t_3^{6k} \rangle$
Γ_8:	$\mathbb{Z}_{36k+2}$	$N_8 = \langle t_1, t_2, t_3^{6k+2} \rangle$
Γ_9:	$\mathbb{Z}_{36k+24}$	$N_9 = \langle t_1, t_2, t_3^{6k+4} \rangle$

$$\Gamma_1 = \langle t_1, t_2, t_3, \alpha \mid [t_1, t_2] = t_3^{4k},\ [t_3, t_2] = [t_3, t_1] = 1,\ \alpha^4 = t_3,$$
$$\alpha t_1 \alpha^{-1} = t_2^{-1},\ \alpha t_2 \alpha^{-1} = t_1,\ [\alpha, t_3] = 1\rangle$$
$$\Gamma_2 = \langle t_1, t_2, t_3, \alpha \mid [t_1, t_2] = t_3^{4k},\ [t_3, t_1] = [t_3, t_2] = 1,\ \alpha^4 = t_3^3,$$
$$\alpha t_1 \alpha^{-1} = t_2^{-1},\ \alpha t_2 \alpha^{-1} = t_1,\ [\alpha, t_3] = 1\rangle$$
$$\Gamma_3 = \langle t_1, t_2, t_3, \alpha \mid [t_1, t_2] = t_3^{3k},\ \alpha^3 = t_3^2,\ v\alpha t_2 \alpha^{-1} = t_1,$$
$$\alpha t_1 \alpha^{-1} = t_2^{-1} t_1^{-1}\rangle$$
$$\Gamma_4 = \langle t_1, t_2, t_3, \alpha \mid [t_1, t_2] = t_3^{3k-1},\ \alpha^3 = t_3,\ \alpha t_2 \alpha^{-1} = t_1,$$
$$\alpha t_1 \alpha^{-1} = t_2^{-1} t_1^{-1}\rangle$$
$$\Gamma_5 = \langle t_1, t_2, t_3, \alpha \mid [t_1, t_2] = t_3^{3k-2},\ \alpha^3 = t_3^2,\ \alpha t_2 \alpha^{-1} = t_1,$$
$$\alpha t_1 \alpha^{-1} = t_2^{-1} t_1^{-1}\rangle$$
$$\Gamma_6 = \langle t_1, t_2, t_3, \alpha \mid [t_1, t_2] = t_3^{6k},\ \ \alpha t_2 \alpha^{-1} = t_2 t_1,\ \ \alpha t_1 \alpha^{-1} = t_2^{-1},$$
$$\alpha^6 = t_3\rangle$$
$$\Gamma_7 = \langle t_1, t_2, t_3, \alpha \mid [t_1, t_2] = t_3^{6k},\ \ \alpha t_2 \alpha^{-1} = t_2 t_1,\ \ \alpha t_1 \alpha^{-1} = t_2^{-1},$$
$$\alpha^6 = t_3^5\rangle$$
$$\Gamma_8 = \langle t_1, t_2, t_3, \alpha \mid [t_1, t_2] = t_3^{6k+2},\ \ \alpha t_2 \alpha^{-1} = t_2 t_1,$$
$$\alpha t_1 \alpha^{-1} = t_2^{-1},\ \ \alpha^6 = t_3^5\rangle$$
$$\Gamma_9 = \langle t_1, t_2, t_3, \alpha \mid [t_1, t_2] = t_3^{6k+4},\ \ \alpha t_2 \alpha^{-1} = t_2 t_1,$$
$$\alpha t_1 \alpha^{-1} = t_2^{-1},\ \ \alpha^6 = t_3\rangle$$

$f_1(x, y, z) = (y, -x, z - xy + \frac{1}{16k})$.
$f_2(x, y, z) = (-y, x, z - xy + \frac{1}{16k})$.
$f_3(x, y, z) = (-y, x - y - \frac{1}{2}, \frac{1}{9k} - \frac{1}{24} + z + \frac{1}{2}y^2 - xy)$.
$f_4(x, y, z) = (-x + y + \frac{1}{2}, -x, z + \frac{1}{2}x^2 - xy + \frac{1}{9k-3} + \frac{1}{24} - \frac{x}{2})$.
$f_5(x, y, z) = (-y, x - y - \frac{1}{2}, z + \frac{1}{2}y^2 - xy + \frac{1}{9k-6} - \frac{1}{24})$.
$f_6(x, y, z) = (y, -x + y + \frac{1}{2}, z + \frac{1}{2}y^2 - xy + \frac{1}{36k} + \frac{1}{8})$.
$f_7(x, y, z) = (x - y + \frac{1}{2}, x, z + \frac{1}{2}x^2 - xy + \frac{1}{36k} - \frac{1}{8})$.
$f_8(x, y, z) = (x - y + \frac{1}{2}, x, z + \frac{1}{2}x^2 - xy + \frac{1}{36k} - \frac{1}{8})$.
$f_9(x, y, z) = (y, -x + y + \frac{1}{2}, z + \frac{1}{2}y^2 - xy + \frac{1}{36k} + \frac{1}{8})$.

Proof. We shall deal with only Γ_1. Let N be a normal nilpotent subgroup of Γ_1 such that $G = \Gamma_1/N$ is abelian. Then

$$[\Gamma_1, \Gamma_1] = \langle t_1 t_2, t_2^2, t_3^{4k} \rangle \subset N \subset \Gamma_1.$$

Suppose N contains both $t_1 t_2^{\ell_1} t_3^{\ell_2}, t_2 t_3^m$. Then N can be represented by an ordered set of generators

$$\langle t_1 t_2^{\ell_1} t_3^{\ell_2}, t_2 t_3^m, t_3^n \rangle$$

Then N reduces to $\langle t_1 t_3^{\ell}, t_2 t_3^m, t_3^n \rangle$. Since N contains $t_1 t_2$ and t_2^2, we have $t_3^{2m}, t_3^{\ell+m} \in N$. Thus $2m$ and $\ell + m$ must be multiples of n.

Note that $0 \leq \ell, m < n$. Thus $\ell = 0, \frac{n}{2}$ and $m = 0, \frac{n}{2}$. Since

$$[t_1 t_3^{\ell},\ t_2 t_3^m] = [t_1,\ t_2] = t_3^{4k} \in N,$$

$4k$ must be a multiple of n. Note that we shall do only an infra-nilmanifold M_1 case, i.e., $n = 4k$. Suppose N does not contain either $t_1 t_2^{\ell_1} t_3^{\ell_2}$ or $t_2 t_3^m$. Since G is not abelian in these cases, we induce a contradiction. So, the possible normal nilpotent subgroups are

$$N_1 = \langle t_1,\ t_2,\ t_3^{4k} \rangle, \quad N_2 = \langle t_1 t_3^{2k},\ t_2 t_3^{2k},\ t_3^{4k} \rangle.$$

It is not hard to see $N_1 \underset{R}{\sim} N_2$ by using $\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$. Thus we get

$$N = N_1 = \langle t_1,\ t_2,\ t_3^{4k} \rangle.$$

Therefore there exists only one $\mathbb{Z}_{16k} = \Gamma_1/N$ free action on the nilmanifold $\mathcal{H}/N$ which yields an infra-nilmanifold homeomorphic to $\mathcal{H}/\Gamma_1$.

Suppose N does not contain either $t_1 t_2^{\ell_1} t_3^{\ell_2}$ or $t_2 t_3^m$. Since G is not abelian in these cases, we induce a contradiction.

Let $N = \langle t_1,\ t_2,\ \alpha^{16k}\rangle$. Since G is generated by the images of α, it is enough to calculate conjugations of α by (I, B), where $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Aut}(\mathcal{H})$.

For $\alpha = (a, A) = \left(\begin{bmatrix} 1 & 0 & \frac{1}{16k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$, we have

$$(I, B)(a, A)(I, B)^{-1} = (B(a), A) = \left(\begin{bmatrix} 1 & 0 & \frac{1}{16k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A \right).$$

It acts on $\mathcal{H}$ by

$$\left(\begin{bmatrix} 1 & 0 & \frac{1}{16k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A \right) \cdot \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y & z - xy + \frac{1}{16k} \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore if $f : \mathcal{H} \to \mathcal{H}$ is the map generated by α, then

$$f(x, y, z) = (y, -x, z - xy + \frac{1}{16k}).$$

The other cases can be done by using a similar method. □

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