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I-RINGS AND TRIANGULAR MATRIX RINGS

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ABSTRACT. All rings are assumed to be associative but do not necessarily have an identity. In this paper, we carry out a study of ring theoretic properties of formal triangular matrix rings. Some results are obtained on these rings concerning properties such as being I_0-ring, I-ring, exchange ring.

Throughout this paper all rings are assumed to be associative but do not necessarily have an identity. When a ring has an identity, modules are assumed to be unital. The Jacobson radical of a ring R will be denoted by $J(R)$.

Let R and S be rings and ${}_RV_S$ a left R right S bimodule. The formal triangular matrix ring $U = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ has as its elements formal matrices $\begin{bmatrix} r & v \\ 0 & s \end{bmatrix}$ where $r \in R, s \in S$ and $v \in V$ with addition defined coordinatewise and multiplication given by $\begin{bmatrix} r & v \\ 0 & s \end{bmatrix} \begin{bmatrix} r' & v' \\ 0 & s' \end{bmatrix} = \begin{bmatrix} rr' & rv' + vs' \\ 0 & ss' \end{bmatrix}$.

LEMMA 1. [4] *If R is a ring, the following conditions are equivalent:*

(1) *Every left ideal $L \nsubseteq J(R)$ contains a nonzero idempotent.*

(2) *Every right ideal $L \nsubseteq J(R)$ contains a nonzero idempotent.*

(3) *If $a \notin J(R)$, then $xax = x$ for some $x \neq 0$.*

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Proof. Given (1), let $a \notin J(R)$. Let L be the left ideal of R generated by a. Every element of L has the form $ra + na$ where $n \in \mathbb{Z}$ and $r \in R$. Let $e \in L$ be a nonzero idempotent in L. Hence $e = ra$ for some $r \in R$. (3) follows with $x = rar$. Conversely let $L \nsubseteq J(R)$ be a left ideal. There exists $a \in L$ and $a \notin J(R)$. By (3), $xax = x$ for some $x \neq 0$. $xaxa = xa \in L$ is a nonzero idempotent. The proof that (2) $\Leftrightarrow$ (3) is analogous. $\square$

DEFINITION 2. [4] *A ring R is called an $I_0 - ring$ if it satisfies the conditions of lemma 1. An I_0-ring in which idempotents can be lifted modulo $J(R)$ is called an I-ring.*

The class of I-rings is quite large. It obviously contains all division rings, and, more generally, contains all local rings, where in this paper, a ring R will be called local if it has an identity and $R/J(R)$ is a division ring.

PROPOSITION 3. [4] *Let R be a ring in which idempotents can be lifted modulo $J(R)$. Then R is an I-ring if and only if $R/J(R)$ is an I-ring.*

Let R be a ring and I be a right ideal in R. I is modular if there exists $e \in R$ such that

$$r - er \in I \quad \text{for all } r \in R \quad [3].$$

It is well known that $J(R)$ is the intersection of all modular maximal right ideals of R.

PROPOSITION 4. *The set of modular maximal right ideals of U is given by $\{\begin{bmatrix} I & {}_RV_S \\ 0 & K \end{bmatrix} \mid$ either $I = R$ and K is a modular maximal* *right ideal of S or I is a modular maximal right ideal of R and $K = S$}*

Proof. Let $\begin{bmatrix} I & {}_RV_S \\ 0 & K \end{bmatrix}$ be a modular maximal right ideal of U. Clearly I is a modular right ideal of R and K is a modular right ideal of S. If $K \neq S$, then choosing a modular maximal right ideal of K' of S with $K \subseteq K'$, we see that $\begin{bmatrix} R & V \\ 0 & K' \end{bmatrix}$ is a modular maximal right ideal of U and $\begin{bmatrix} I & {}_RV_S \\ 0 & K \end{bmatrix} \leq \begin{bmatrix} R & V \\ 0 & K' \end{bmatrix}$. The maximality of $\begin{bmatrix} I & {}_RV_S \\ 0 & K \end{bmatrix}$ yields $I = R$ and $K = K'$. If, on the other hand, $K = S$, then $\begin{bmatrix} I & V \\ 0 & S \end{bmatrix}$ is a modular maximal right ideal of V. Hence I is a modular maximal right ideal of R. Thus any modular maximal right ideal of U has to be either $\begin{bmatrix} R & V \\ 0 & K \end{bmatrix}$ with K a modular maximal right ideal of S or $\begin{bmatrix} I & V \\ 0 & S \end{bmatrix}$ with I a modular maximal right ideal of R.

Conversely, right ideals of the above form are clearly modular maximal right ideals of U. □

COROLLARY 5. $J(U) = \begin{bmatrix} J(R) & V \\ 0 & J(S) \end{bmatrix}$.

Proof. This is an immediate consequence of Proposition 4. □

COROLLARY 6.

(1) $\begin{bmatrix} r & v \\ 0 & s \end{bmatrix} + J(U) \longrightarrow (r + J(R), s + J(S))$ *is a ring isomorphism of $U/J(U)$ with $R/J(R) \times S/J(S)$.*

(2) *Idempotents mod $J(U)$ can be lifted in U if and only if idempotents mod $J(R)$ can be lifted in R and idempotents mod $J(S)$ can be lifted in S.*

THEOREM 7. $U = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ *is an I_0-ring if and only if R and S are I_0-rings.*

Proof. Assume that U is an I_0-ring. Let I be a right ideal of R not contained in $J(R)$. $\bar{I} = \begin{bmatrix} I & IV \\ 0 & 0 \end{bmatrix}$ is a right ideal of U with $\bar{I} \nsubseteq J(U)$. Hence there exists an element $\begin{bmatrix} e & v \\ 0 & 0 \end{bmatrix} \in \bar{I}$ with $\begin{bmatrix} e & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e & v \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} e & v \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. $\begin{bmatrix} e & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e^2 & ev \\ 0 & 0 \end{bmatrix}$. Hence $e^2 = e$ and $ev = v$. We see that either $e \neq$ or $v \neq 0$. If $e = 0$, then $ev = v = 0$. We conclude that $e \neq 0$. Thus $e \in I$ and $e^2 = e$ in R. Thus I contains a nonzero idempotent. Let K be a right ideal of S such that $K \nsubseteq J(S)$. $\bar{K} = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix}$ is a right ideal of V and $\bar{K} \nsubseteq J(U)$. There exists an element $0 \neq f \in K$ with

$$\begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}, \quad f = f^2 .$$

K contains a nonzero idempotent. Therefore R and S are I_0-rings.

Conversely, assume that R and S are I_0-rings. Let $\bar{I}$ be a right ideal of U with $\bar{I} \nsubseteq J(U)$. Then there exists an element $\begin{bmatrix} a & v \\ 0 & b \end{bmatrix} \in \bar{I}$ with $\begin{bmatrix} a & v \\ 0 & b \end{bmatrix} \notin J(U)$. This means $a \notin J(R)$ or $b \notin J(S)$.

To show that $\bar{I}$ contains a nonzero idempotent, it suffices to show that $\begin{bmatrix} a & v \\ 0 & b \end{bmatrix} U = \begin{bmatrix} a & v \\ 0 & b \end{bmatrix} \begin{bmatrix} R & V \\ 0 & S \end{bmatrix} = \begin{bmatrix} aR & aV + vS \\ 0 & bS \end{bmatrix}$ contains a nonzero idempotent.

If $b \notin J(S)$, then $xbx = x$ for some $x \neq 0$ by Lemma 1. $(bx)^2 = bx \in bS$. Let $bx = f$. $f \in bS$ and $\begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}$ is a nonzero idempotent in uU where $u = \begin{bmatrix} a & v \\ 0 & b \end{bmatrix}$.

Suppose $a \notin J(R)$. Again by Lemma 1, there exists a nonzero idempotent $e \in R$ of the form $e = ax$ for some $x \in R$. $xax = x \neq 0$ implies $(ax)^2 = e^2 = e$.

$$\begin{bmatrix} e & ev \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ax & axv \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & v \\ 0 & b \end{bmatrix} \begin{bmatrix} x & xv \\ 0 & 0 \end{bmatrix} \in uU$$

$$\text{and } \begin{bmatrix} ax & axv \\ 0 & 0 \end{bmatrix} \begin{bmatrix} ax & axv \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ax & axv \\ 0 & 0 \end{bmatrix}$$

This shows that V is an I_0-ring. □

COROLLARY 7. $U = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ *is an I-ring if and only if R and S are I-rings.*

Proof. $J(U) = \begin{bmatrix} J(R) & V \\ 0 & J(S) \end{bmatrix}$ by corollary 5. By corollary 6, idempotent mod $J(U)$ can be lifted in U if and only if idempotent mod $J(R)$ can be lifted in R and idempotent mod $J(S)$ can be lifted in S.

By theorem 7, U is an I_0-ring if and only if R and S are I_0-rings. □

An associative unital ring R is said to be an exchange ring if R_R has the exahange property introduced by Crawlery and Jó nsson[2][5].

If R is a ring without identity, we denote R' the unitalization of R ; that is, $R' = R \oplus \mathbb{Z}$ with addition and multiplication defined by $(x,n)+(y,m) = (x+y, n+m)$, and $(x,n)(y,m) = (xy+ny+mx, nm)$ for all $x, y \in R$ and $n, m \in \mathbb{Z}$.

LEMMA 8. [1] *Let R a ring without unity and let T be a unital ring containing R as a (two-sided) ideal. Then the following conditions are equivalent for an element $x \in R$;*

(1) There exists $e^2 = e \in R$ with $e - x \in T(x - x^2)$.

(2) There exists $e^2 = e \in Rx$ with $c \in T$ such that $(1-e)-c(1-x) \in J(R)$.

(3) There exists $e^2 = e \in Rx$ such that $T = Re + T(1-x)$.

(4) There exists $e^2 = e \in Rx$ such that $1 - e \in T(1-x)$.

(5) There exists $r, s \in R$ and $e^2 = e \in R$ such that $e = rx = s + x - sx$.

Let I be a ring without unity and let R be a unital ring containing I as an ideal. Let M_S be a module over a unital ring S such that there exists an isomorphism $\phi : \mathrm{End}(M_S) \to R$. Let $X = M \oplus Y = N_1 \oplus N_2$ (*) be two decomposition of a right S-module X. Denote by π the idempotent in $\mathrm{End}(X)$ with image M and kernel Y, and identify $\mathrm{End}(M)$ with $\pi\mathrm{End}(X)\pi$. We say that (*) is I-admissible if $\pi\tau_2\pi \in \varphi^{-1}(I)$ where $\tau_i \in \mathrm{End}(X)$ is the projection onto N_i determined by the decomposition $X = N_1 \oplus N_2$.

THEOREM 9. [1] *Let (I, R) and $\phi : End(M_S) \to R$ be as above. Then the following conditions are equivalent :*

(a) For all $x \in I$, there exist $e = e^2 \in I$ and $r, s \in I$ such that $e = rx = x + s - sx$.

(b) For all I-admissible decompositions $X = M \oplus Y = N_1 \oplus N_2$, there exist $N_i' \subseteq N_i$ such that $X = M \oplus N_1' \oplus N_2'$.

(c) For all $x \in I$, there exist $e = e^2 \in I$ and $r, s \in I$ such that $e = xr = s + x - xs$.

DEFINITION 10. *A ring without unity I is called an exchange ring if it satisfies the equivalent conditions in Theorem 9.*

Note that I being an exchange ring is a symmetric condition which does not depend on the particular unital ring where I is embedded as an ideal.

EXAMPLE. [1] *(1) If I is an ideal of a unital exchange ring, then I is an exchange ring. To see this, take an element $x \in I$. There exist $r, s \in R$ such that $e = xr$ and $1 - e = (1 - x)(1 - s)$. Clearly $e \in I$ and therefore $s = e - x + xs \in I$. Consequently we can write $e = x(re) = x + s - xs$, with $re, s \in I$ and condition (c) of Theorem 9 is satisfied.*

(2) The radical rings [3] are exactly the exchange rings without nonzero idempotents.

(3) A ring I is said to be π-regular if for all $x \in I$, there exist a positive integer n and $y \in I$ such that $x^n = x^n y x^n$. All the (nonunital) π-regular rings are exchange rings.

THEOREM 11. [1] *Let I be an ideal of a (possibly nonunital) ring L. Then L is an exchange ring if and only if I and L/I are exchange rings and idempotents can be lifted modulo I.*

THEOREM 12. *$U = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ is an exchange ring if and only if R and S are exchange rings.*

Proof. Assume that R and S are exchange rings.

$$J(U) = \begin{bmatrix} J(R) & V \\ 0 & J(S) \end{bmatrix}$$

is a radical ring. By example (2), $J(U)$ is an exchange ring.

$V/J(V) = \begin{bmatrix} R/J(R) & 0 \\ 0 & S/J(S) \end{bmatrix}$ is an exchange ring.

Conversely assume that U is an exchange ring. $U/J(U)$ is an exchange ring and $U/J(U) \cong R/J(R) \times S/J(S)$. Therefore $R/J(R)$ and $S/J(S)$ are exchange rings. By Theorem 11 and example (2), R and S are exchange rings. □

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