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SOME REMARKS ON H_v-GROUPS

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ABSTRACT. Vogiouklis introduced H_v-hyperstructures and gave the "open problem: for H_v-groups, we have $\beta^* = \beta$". We have an affirmative result about this open problem for some special cases.

We study β relations on H_v-quasigroups. When a set H has at least three elements and $(H, \cdot)$ is an H_v-quasigroup with a weak scalar e, if there are elements $x, y \in H$ such that $xy = H \setminus \{e\}$, then we have $(xy)(xy) = H$.

1. Introduction

In 1934 S. F. Marty[?] introduced a basic concept of hyperstructures and, in 1990 T. Vougioklis[?] introduced H_v-hyperstructures and studied about H_v-groups defined on the same set $H = \{e, a, b\}$.

Freni[?] proved that, if H is a hypergroup, then $\beta^* = \beta$. Vogiouklis[?, Thm.1.2.2.] proved that, when $(H, \cdot)$ is an H_v-group, the fundamental relation β^* is the transitive closure of the relation β. And also he gave the following "open problem".

$$\text{For } H_v\text{-groups, we have } \beta^* = \beta.$$

We have an affirmative result about this open problem for some special cases.

We study β relations on H_v-quasigroups. When a set H has at least three elements and $(H, \cdot)$ is an H_v-quasigroup with a weak scalar e, if there are elements $x, y \in H$ such that $xy = H \setminus \{e\}$, then we have

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$(xy)(xy) = H$. Using this lemma, when $u_1 = H \setminus \{a\}$ and $u_2 = H \setminus K$ where $u_1,\ u_2 \in \mathcal{U}$, $a \in H \setminus \{e\}$, and $K \subset H \setminus \{e\}$, if $u_1 \cup u_2 = H$, then β is an equivalence relation and $H/\beta \cong \mathbb{Z}_1$.

In Theorem ??, let $u_1 = H \setminus H_1$ and $u_2 = H \setminus K$ where $u_1,\ u_2 \in \mathcal{U}$, $e \in u_1 \cap u_2$. If $u_1 \cup u_2 = H$, then also β is an equivalence relation and $H/\beta \cong \mathbb{Z}_1$.

2. Some results on H_v-groups

Let H be a nonempty set and $P^*(H)$ be the set of all nonempty subsets of H. We say that a function $\cdot : H \times H \longrightarrow P^*(H)$ is a hyperoperation on H. We assume that for subsets $A,\ B$ of H

$$A \cdot B = \bigcup_{a \in A,\ b \in B} a \cdot b.$$

A hyperoperation $\cdot$ on H is *weak associative*(WASS) if the sets $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$ have nonempty intersection, i.e.

$$(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset \quad \text{for all } x,\ y,\ z \text{ in } H.$$

We say that a hyperoperation $\cdot$ on H satisfies the *reproduction property* if

$$x \cdot H = H \cdot x = H \quad \text{for all } x \text{ in } H.$$

We call an ordered pair $(H,\ \cdot\ )$ as a *hyperstructure* on H. A hyperstructure $(H,\ \cdot\ )$ is called an *H_v-semigroup* if $\cdot$ is WASS and an *H_v-quasigroup* if $\cdot$ satisfies the reproduction property.

Recall that a hyperstructure $(H,\ \cdot\ )$ is called an *H_v-group* if it is an *H_v-quasigroup* and an *H_v-semigroup*. Furthermore, an *H_v-group* $(H,\ \cdot\ )$ is called a *hypergroup* if $\cdot$ is associative.

From now on, we do not use the notation $\cdot$ in the equation if it does not cause any confusion.

DEFINITION 2.1. Let $(H, \cdot)$ be a hyperstructure.

An element $s \in H$ is called a *weak scalar* if sx and xs are singletons for all $x \in H$. Furthermore, if $sx = xs$, then s is called a *scalar.*

An element $u \in H$ is called a *unit* if $x \in ux \cap xu$ for all $x \in H$.

An element $e \in H$ is called a *scalar unit* if $ex = xe = \{x\}$ for all $x \in H$.

Note that if a hyperstructure $(H, \cdot)$ has a scalar unit, then it is unique.

EXAMPLE 2.2. Let H be a set $\{e, a, b\}$. Then there is an H_v-quasigroup with a weak scalar but not scalar e.

$\cdot$	e	a	b
e	e	a	b
a	b	H	H
b	a	H	H

Let $(H, \cdot)$ and $(H', *)$ be two hyperstructures. A function $f : (H, \cdot) \to (H', *)$ is an *isomorphism* if $f(x \cdot y) = f(x) * f(y)$ for all $x, y \in H$ and it is a bijection. We say that H is *isomorphic* to H' if there is an isomorphism between $(H, \cdot)$ and $(H', *)$, and we denote it by $H \cong H'$.

Let H be a hyperstructure. Let us denote by $\mathcal{U}$ the set of all finite products of elements of H and define the relation β in H as follows :

$$x\beta y \quad \text{iff} \quad \{x, y\} \subset u \quad \text{for some} \quad u \in \mathcal{U}.$$

In general, β is not an equivalence relation. For example, let H be the set $\{e, a, b\}$ and consider following H_v-structure:

$\cdot$	e	a	b
e	e	e	e
a	e	$\{e, a\}$	e
b	e	e	$\{e, b\}$

Then the β relation on H satisfies the reflexive and symmetric laws but not the transitive law. For instance, we have $a\beta e$ and $e\beta b$ but not $a\beta b$.

Let $(H, \cdot)$ be an H_v-group. We define the β^* relation as the smallest equivalence relation, one can say also congruence, such that the quotient H/β^* is a group. The β^* is called *fundamental equivalence relation.*

Freni[?] proved that, if H is a hypergroup, then $\beta^* = \beta$. Vougiouklis [?, Thm.1.2.2.] proved that, when $(H, \cdot)$ is an H_v-group, the fundamental relation β^* is the transitive closure of the relation β. And also he mentioned the following "open problem".

$$\text{For } H_v\text{-groups, we have } \beta^* = \beta.$$

We study this open problem for some special cases.

From now on, a set H has at least three elements and $(H, \cdot)$ is an H_v-quasigroup with a weak scalar e. We have some results for H_v-quasigroup.

LEMMA 2.3. *If there are elements $x,\ y \in H$ such that $xy = H \setminus \{e\}$, then $(xy)(xy) = H$.*

In particular, if there is a finite product u of elements of H containing $H \setminus \{e\}$, then we have $u^2 = H$.

Proof. Assume that $xy = H \setminus \{e\}$ for some $x, y \in H$. Since H is an H_v-quasigroup and e is a weak scalar, there are $a,\ b \in H \setminus \{e\}$ such that $ae \cap be = \emptyset$. Then we have

$$\begin{aligned}(xy)(xy) &= (H \setminus \{e\})(H \setminus \{e\}) \\ &\supset a(H \setminus \{e\}) \cup b(H \setminus \{e\}) \\ &\supset (H \setminus ae) \cup (H \setminus be) \\ &= H \setminus (ae \cap be) \\ &= H.\end{aligned}$$

Therefore, we have $(xy)(xy) = H$. □

When H is an H_v-quasigroup, if there is $x, y \in H$ such that $xy = H$, then clearly β is an equivalence relation and $H/\beta \cong \mathbb{Z}_1$.

LEMMA 2.4. *Let $u_1 = H \setminus \{a\}$ and $u_2 = H \setminus K$ where $u_1,\ u_2 \in \mathcal{U}$ and $e \in u_1 \cap u_2$. If $u_1 \cup u_2 = H$, then β is an equivalence relation and $H/\beta \cong \mathbb{Z}_1$.*

Proof. By the assumption, we have

$$\begin{aligned}u_1u_2 &\supset (H \setminus \{a\})(H \setminus K) \\ &\supset e(H \setminus K) \cup (H \setminus \{a\})e \\ &\supset (H \setminus eK) \cup (H \setminus ae) \\ &= H \setminus (eK \cap ae).\end{aligned}$$

We may assume that $u_1u_2 \neq H$. Then $eK \cap ae \neq \emptyset$ and so $eK \cap ae = ae$, since e is a weak scalar. Hence we have $u_1u_2 = H \setminus ae$. By the hypothesis, $K \cap \{a\} = \emptyset$ and $a \in u_2$. Hence we have $u_1u_2 \supset ea$ and $ae \neq ea$.

Therefore we have

$$\begin{aligned} u_1u_1 &\supset (H \setminus \{a\})(H \setminus \{a\}) \\ &\supset e(H \setminus \{a\}) \cup (H \setminus \{a\})e \\ &\supset (H \setminus ea) \cup (H \setminus ae) \\ &= H \setminus (ea \cap ae) \\ &= H. \end{aligned}$$

Hence we have $H/\beta \cong \mathbb{Z}_1$. □

THEOREM 2.5. *Let H_1 and H_2 be subsets of H where H_1 is a nonempty finite set and let $u_1 = H \setminus H_1$ and $u_2 = H \setminus H_2$ for some $u_1,\ u_2 \in \mathcal{U}$ and $e \in u_1 \cap u_2$. If $u_1 \cup u_2 = H$, then β is an equivalence relation and $H/\beta \cong \mathbb{Z}_1$.*

Proof. We proceed by induction on the number of elements of the set H_1. In case $|H_1| = 1$, it follows from Lemma ??.

Suppose that it is true for $|H_1| < n$.

Let $|H_1| = n$. Then we have $|H_1e|,\ |eH_1| \leq n$.

Since $u_1 \cup u_2 = H$, we have $H_1 \cap H_2 = \emptyset$. Hence, by the assumption, we have

$$\begin{aligned} u_1u_2 &\supset (H \setminus H_1)(H \setminus H_2) \\ &= [(H \setminus H_1) \cup H_2][(H \setminus H_2) \cup H_1] \\ &\supset e[(H \setminus H_2) \cup H_1] \cup [(H \setminus H_1) \cup H_2]e \\ &\supset [(H \setminus eH_2) \cup eH_1] \cup [(H \setminus H_1e) \cup H_2e] \\ &= [(H \setminus eH_2) \cup (H \setminus H_1e)] \cup eH_1 \cup H_2e \\ &= [H \setminus (eH_2 \cap H_1e)] \cup eH_1 \cup H_2e \\ &= H \setminus [(eH_2 \cap H_1e) \setminus (eH_1 \cup H_2e)]. \end{aligned}$$

On the other hand, let

$$K_{11} = (eH_1 \cap H_1e) \setminus (eH_2 \cup H_2e)$$
$$K_{12} = (eH_2 \cap H_1e) \setminus (eH_1 \cup H_2e)$$
$$K_{21} = (eH_1 \cap H_2e) \setminus (eH_2 \cup H_1e)$$
$$K_{22} = (eH_2 \cap H_2e) \setminus (eH_1 \cup H_1e).$$

Then we have, $u_1u_2 \supset H \setminus K_{12}$ and similarly we have

$$u_1u_1 \supset H \setminus K_{11}, \quad u_2u_1 \supset H \setminus K_{21}, \quad \text{and} \quad u_2u_2 \supset H \setminus K_{22}.$$

Note that

$$\begin{aligned} K_{12} \cap K_{11} &= [(eH_2 \cap H_1e) \setminus (eH_1 \cup H_2e)] \cap [(eH_1 \cap H_1e) \setminus (eH_2 \cup H_2e)] \\ &= (eH_2 \cap H_1e \cap eH_1 \cap H_1e) \setminus (eH_1 \cup H_2e \cup eH_2 \cup H_2e) \\ &= \emptyset, \end{aligned}$$

$$\begin{aligned} u_1u_1 &\supset (H \setminus H_1)(H \setminus H_1) \\ &\supset e(H \setminus H_1) \\ &= e[(H \setminus H_1) \cup H_2] \\ &\supset (H \setminus eH_1) \cup eH_2, \end{aligned}$$

and

$$\begin{aligned} u_1u_2 &\supset (H \setminus H_1)(H \setminus H_2) \\ &\supset e(H \setminus H_2) \\ &= e[(H \setminus H_2) \cup H_1] \\ &\supset (H \setminus eH_2) \cup eH_1. \end{aligned}$$

Hence we have $u_1u_1 \cup u_1u_2 \supset [(H \setminus eH_1) \cup eH_2] \cup [(H \setminus eH_2) \cup eH_1] = H$.

Using the same method, we have

$$K_{11} \cap K_{21} = K_{11} \cap K_{22} = K_{12} \cap K_{21} = K_{12} \cap K_{22} = K_{21} \cap K_{22} = \emptyset$$

and

$$u_1u_1 \cup u_2u_1 = u_1u_1 \cup u_2u_2 = u_1u_2 \cup u_2u_1 = u_1u_2 \cup u_2u_2 = u_2u_1 \cup u_2u_2 = H.$$

(Case 1) $|K_{12}| = n$.

Since $|H_1 e| \leq n$, we have $K_{12} = (eH_2 \cap H_1 e) \setminus (eH_1 \cup H_2 e) = H_1 e$ and so $H_1 e \subset eH_2$. Hence $K_{11} = (eH_1 \cap H_1 e) \setminus (eH_2 \cup H_2 e) = \emptyset$ and so $u_1 u_1 \supset H \setminus K_{11} = H$.

(Case 2) $|K_{12}| < n$.

Subcase 2-1) $e \in K_{12}$.

Since $K_{12} \cap K_{11} = \emptyset$, we have $e \notin K_{11}$. Hence we have $e \in u_1 u_1$. On the other hand, since $K_{12} \cap K_{21} = \emptyset$, we have $e \notin K_{21}$. Hence we have $e \in u_2 u_1$. Therefore, $e \in u_1 u_1 \cap u_2 u_1$ and $u_1 u_1 \cup u_2 u_1 = H$. If $|K_{11}| = n$, then $|eH_1 \cap H_1 e| = n$, since $|H_1 e|$, $|eH_1| \leq n$. Hence we have $eH_1 = H_1 e$ and so $K_{12} = \emptyset$. Therefore $u_1 u_2 \supset H \setminus K_{12} = H$. If $|K_{11}| < n$, by induction hypothesis, we have $H/\beta \cong \mathbb{Z}_1$.

Subcase 2-2) $e \notin K_{12}$.

Then we have $e \in u_1 u_2$. Since $u_1 u_1 \cup u_2 u_2 = H$, we have $e \in u_2 u_2$ or $e \in u_1 u_1$. Therefore, $e \in u_1 u_2 \cap u_2 u_2$ or $e \in u_1 u_2 \cap u_1 u_1$. Since $u_1 u_2 \cup u_2 u_2 = u_1 u_2 \cup u_1 u_1 = H$, by induction hypothesis, we have $H/\beta \cong \mathbb{Z}_1$. □

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