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STABILITY OF SOME GENERALIZED QUADRATIC FUNCTIONAL EQUATION

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ABSTRACT. In this paper we prove the stability of some generaliged quadratic functional equation

$$a^2 f\left(\frac{x+y}{a}\right) + a^2 f\left(\frac{x-y}{a}\right) = 2f(x) + 2f(y).$$

In 1940, S. M. Ulam ([9]) posed the following question on the stability of homomorphisms: Given a metric group $(G, +, d)$, a number $\epsilon > 0$ and a mapping $f : G \to G$ which satisfies the inequality

$$d(f(x+y), f(x) + f(y)) < \epsilon$$

for all $x, y \in G$, does there exist an automorphism $a : G \to G$ and a constant $k > 0$, depending only on G, such that for all $x \in G$

$$d(f(x), a(x)) < k\epsilon?$$

This question became a source of the stability theory in the Hyers-Ulam sense.

The case of approximately additive mappings was solved by D. H. Hyers ([1]) under the assumption that X and Y are Banach spaces. Later, many authors proved numerous theorems for the stability of functional equations.

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The quadratic function $f(x) = x^2$ is a solution of the functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$. So, every solution of the functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is said to be a quadratic function. In this paper we deal with a generalized quadratic functional equation $a^2 f\left(\frac{x+y}{a}\right) + a^2 f\left(\frac{x-y}{a}\right) = 2f(x) + 2f(y)$, where a is a nonzero real constant.

Let $\mathbb{R}_+$ denote the set of nonnegative real numbers. Recall that a function $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is homogeneous of degree $p > 0$ if it satisfies $H(tu, tv) = t^p H(u, v)$ for all nonnegative real numbers t, u and v. Throughout this paper X and Y will be a real normed linear space and a real Banach space, respectively. We may assume that H is homogeneous of degree p. Given a function $f : X \to Y$, we set

$$Df(x,y) := a^2 f\left(\frac{x+y}{a}\right) + a^2 f\left(\frac{x-y}{a}\right) - 2f(x) - 2f(y)$$

for all $x, y \in X$.

THEOREM 1. *Let $\delta \geq 0$ and $0 < p < 2$ be real numbers. If a function $f : X \to Y$ satisfies*

$$||Df(x,y)|| \leq \delta + H(||x||, ||y||) \tag{1}$$

for all $x, y \in X$ and $f(0) = 0$, then there exists a unique quadratic function $Q : X \to Y$ such that

$$||f(x) - Q(x)|| \leq \frac{1}{2}\delta + \frac{1}{4 - 2^p} h(x) \tag{2}$$

for all $x \in X$, where $h(x) = \frac{1}{2} H(||2x||, 0) + H(||x||, ||x||)$.

Proof. Putting $y = 0$ in (1) and replacing x by $2x$ in this result, we have

$$\left|\left| a^2 f\left(\frac{2x}{a}\right) - f(2x) \right|\right| \leq \frac{1}{2}(\delta + H(||2x||, 0)) \tag{3}$$

for all $x \in X$. Putting $y = x$ in (1) we have

$$\left\| a^2 f\left(\frac{2x}{a}\right) - 4f(x) \right\| \le \delta + H(||x||, ||x||) \tag{4}$$

for all $x \in X$. By (3) and (4), we have

$$||f(2x) - 4f(x)|| \le \frac{3}{2}\delta + h(x) \tag{5}$$

for all $x \in X$, where $h(x) = \frac{1}{2}H(||2x||, 0) + H(||x||, ||x||)$. By (5) we have

$$\left\| f(x) - \frac{f(2x)}{4} \right\| \le \frac{3}{8}\delta + \frac{1}{4}h(x) \tag{6}$$

for all $x \in X$. Using (6) we have

$$\begin{aligned} \left\| \frac{f(2^n x)}{4^n} - \frac{f(2^{n+1}x)}{4^{n+1}} \right\| &= \frac{1}{4^n} \left\| f(2^n x) - \frac{f(2 \cdot 2^n x)}{4} \right\| \\ &\le \frac{3}{8} 4^{-n}\delta + \frac{1}{4} 2^{n(p-2)} h(x) \end{aligned} \tag{7}$$

for all $x \in X$ and all positive integers n. From (6) and (7) we have

$$\left\| \frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n} \right\| \le \sum_{k=m}^{n-1} \frac{3}{8} 4^{-k}\delta + \sum_{k=m}^{n-1} \frac{1}{4} 2^{k(p-2)} h(x) \tag{8}$$

for all $x \in X$ and all nonnegative integers m and n with $m < n$. This shows that $\left\{ \frac{f(2^n x)}{4^n} \right\}$ is a Cauchy sequence for all $x \in X$. Consequently, we can define a function $Q : X \to Y$ by

$$Q(x) := \lim_{n \to \infty} \frac{f(2^n x)}{4^n}$$

for all $x \in X$. We have $Q(0) = 0$ and

$$\begin{aligned} ||DQ(x, y)|| &= \lim_{n \to \infty} 4^{-n} ||Df(2^n x, 2^n y)|| \\ &\le \lim_{n \to \infty} (4^{-n}\delta + 2^{(p-2)n} H(||x||, ||y||)) \\ &= 0 \end{aligned}$$

for all $x, y \in X$. Hence

$$a^2Q\left(\frac{x+y}{a}\right) + a^2Q\left(\frac{x-y}{a}\right) = 2Q(x) + 2Q(y) \tag{9}$$

for all $x, y \in X$. Putting $y = 0$ in (9) we have

$$a^2Q\left(\frac{x}{a}\right) = Q(x) \tag{10}$$

for all $x \in X$. Using (9) and (10) we have

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in X$. It follows that Q is quadratic. Putting $m = 0$ in (8) and letting $n \to \infty$ we have (2). Now, let $Q' : X \to Y$ be another quadratic function satisfying (2). Then we have

$$\begin{aligned} ||Q(x) - Q'(x)|| &= 4^{-n}||Q(2^n x) - Q'(2^n x)|| \\ &\le 4^{-n}(||Q(2^n x) - f(2^n x)|| + ||Q'(2^n x) - f(2^n x)||) \\ &\le 4^{-n}\delta + \frac{2}{4-2^p}2^{n(p-2)}h(x) \end{aligned}$$

for all $x \in X$ and all positive integers n. Since

$$\lim_{n\to\infty}\left(4^{-n}\delta + \frac{2}{4-2^p}2^{n(p-2)}h(x)\right) = 0,$$

we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. □ □

THEOREM 2. *Let $2 < p$ be a real number. If a function $f : X \to Y$ satisfies*

$$||Df(x,y)|| \le H(||x||, ||y||) \tag{11}$$

for all $x, y \in X$ and $f(0) = 0$, then there exists a unique quadratic function $Q : X \to Y$ such that

$$||f(x) - Q(x)|| \leq \frac{1}{2^p - 4} h(x) \tag{12}$$

for all $x \in X$, where $h(x) = \frac{1}{2}H(||2x||, 0) + H(||x||, ||x||)$.

Proof. As in the proof of Theorem 1, we see that

$$||f(2x) - 4f(x)|| \leq h(x) \tag{13}$$

for all $x \in X$, where $h(x) = \frac{1}{2}H(||2x||, 0) + H(||x||, ||x||)$. Replacing x by $\frac{x}{2}$ in (13) we have

$$||4f(2^{-1}x) - f(x)|| \leq 2^{-p}h(x) \tag{14}$$

for all $x \in X$. Using (14) we have

$$||4^n f(2^{-n}x) - 4^{n+1} f(2^{-(n+1)}x)|| \leq 2^{-p}2^{n(2-p)}h(x) \tag{15}$$

for all $x \in X$. From (14) and (15) we have

$$||4^n f(2^{-n}x) - f(x)|| \leq \sum_{k=0}^{n-1} 2^{-p}2^{k(2-p)}h(x)$$

for all $x \in X$ and all positive integers n. The rest of the proof is similar to the corresponding part of the case $p < 2$. □ □

THEOREM 3. *Assume that $\delta \geq 0$, $p \in (0, \infty) \setminus \{2\}$, and $\delta + ||(a^2 - 2)f(0)|| = 0$ when $2 < p$. If a function $f : X \to Y$ satisfies (1) for all $x, y \in X$, then there exists a unique quadratic function $Q : X \to Y$ such that*

$$||f(x) - f(0) - Q(x)|| \leq \frac{1}{2}\delta + ||(a^2 - 2)f(0)|| + \frac{1}{|4 - 2^p|} h(x) \tag{16}$$

for all $x \in X$, *where* $h(x) = \frac{1}{2}H(||2x||, 0) + H(||x||, ||x||)$.

Proof. Let $F(x) = f(x) - f(0)$. Then $F(0) = 0$ and

$$\begin{aligned} ||DF(x,y)|| &= ||Df(x,y) - 2(a^2-2)f(0)|| \\ &\le \delta + 2||(a^2-2)f(0)|| + H(||x||, ||y||) \end{aligned}$$

for all $x, y \in X$. Applying Theorem 1 and Theorem 2, we can show that there exists a unique quadratic function $Q : X \to Y$ satisfying (16). □ □

Define a function $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $H(a,b) = (a^p + b^p)\theta$ where $\theta \ge 0$ and $p \in (0, \infty)$. Then H is homogeneous of degree p. Thus we have the following corollary.

COROLLARY 4. *Assume that* $\delta \ge 0$, $p \in (0,\infty)\setminus\{2\}$, *and* $\delta + ||(a^2 - 2)f(0)|| = 0$ *when* $2 < p$. *If a function* $f : X \to Y$ *satisfies*

$$||Df(x,y)|| \le \delta + \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$, *then there exists a unique quadratic function* $Q : X \to Y$ *such that*

$$||f(x) - f(0) - Q(x)|| \le \frac{1}{2}\delta + ||(a^2-2)f(0)|| + \frac{2+2^p}{|4-2^p|}\theta||x||^p$$

for all $x \in X$.

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