미분구적법을 이용한 헬리컬 스프링의 응력해석

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Stress Analysis of Helical Spring Using DQM

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Abstract : DQM(differential quadrature method) is applied to computation of two dimensional elasticity problems in helical springs. Elastic shear stresses in an axially loaded helical spring having rectangular and square cross sections are calculated. The results are compared with those obtained using the method of successive approximations. The differential quadrature method gives good accuracy even when only a limited number of grid points is used.

초 록 : 미분구적법(DQM)을 이용하여 헬리컬 스프링(helical spring)의 2차원적 탄성 문제를 계산하였다. 헬리컬 스프링의 직사각형 및 정사각형 단면적에 축방향 하중(axially loaded)이 작용했을 때의 탄성 전단 응력(elastic shear stress)을 계산하였다. 미분구적법의 결과를 다른 수치해석(successive approximation) 결과와 비교하였으며, 미분구적법은 적은 요소(grid point)를 사용하여 정확한 해석결과를 보여주었다.
Key Words : DQM, helical spring, two demensional elasticity, shear stres

1. Introduction

The problem of determining elastic stresses in helical springs has been treated mathematically by many investigators. The analysis made by Wahl[1] is one of the most widely accepted analyses based on strength-of-materials theory. Wahl[1] used a method analogous to Winkler's method described by Timoshenko[2] for finding bending stresses in curved beams. In solving the differential equations of equilibrium and compatibility by an iterative method, Gohner[3] made probably the first complete and generally applicable elasticity-theory analysis of the stresses using the method of successive approximation.

Here the differential quadrature method(DQM) is applied to two dimensional elasticity problems. Elastic shear stresses in an axially loaded helical springs having rectangular and square cross sections are calculated by DQM. The results are compared with those obtained using the method of successive approximation by Gohner.[3]

2. Governing Differential Equations

Consider a ring sector under the action of two equal and opposite forces P along the axis through the center of the ring and perpendicular to the plane of the ring shown in Fig. 1. This derivation is based on Gohner's analysis of a solid spring. Fig. 2 shows the coordinate system. The basic differential equation is

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{\partial^2 \Phi}{\partial z^2} - \frac{3}{r}\frac{\partial \Phi}{\partial r} + 2c = 0 \qquad (1)$$

where, Φ is the stress function, r is the radius from

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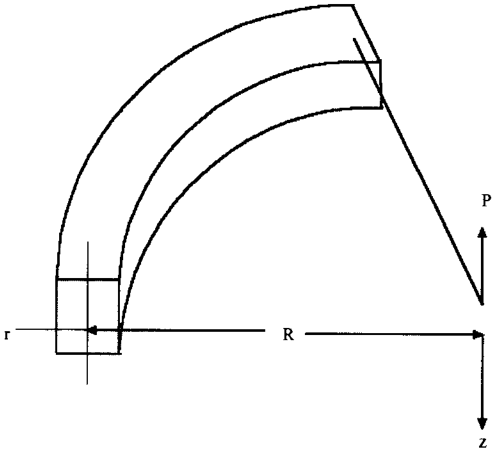

Fig. 1. Ring sector under pure twist

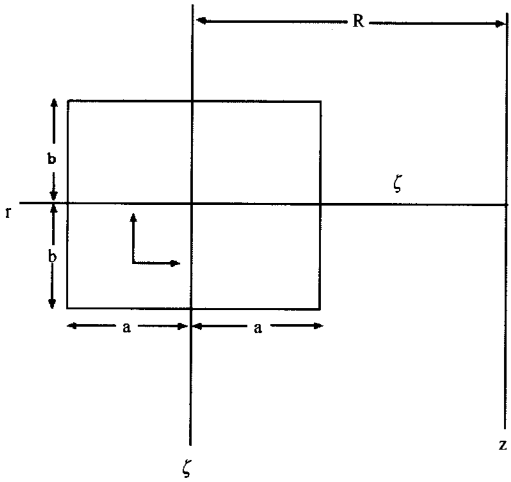

Fig. 2. Coordinate system for ring sector

helix axis to any point in spring, z is the axial position coordinate of any point in the section (see Fig. 1), and c is a constant.

In this case of torsion only the shearing-stress components $\tau_{r\theta}$ and $\tau_{\theta z}$ are different from zero and can be written as

$$\tau_{r\theta}=\frac{-GR^2}{r^2}\frac{\partial\Phi}{\partial z},\qquad \tau_{\theta z}=\frac{GR^2}{r^2}\frac{\partial\Phi}{\partial r} \tag{2}$$

where G is the modulus of rigidity and R is the radius of the ring. The corresponding value of the torque is

$$M_t=-\int\int(\tau_{r\theta}z+\tau_{\theta z}r)\,drdz \tag{3}$$

The unknown constant c can be determined from the torque equation.

Introducing new coordinates ξ and ζ described by Timoshenko and Goodier,[4]

$$\xi=R-r,\qquad \zeta=z \tag{4}$$

Replacing r and z by ξ and ζ, one can rewrite Eq. (1) as

$$\frac{\partial^2\Phi}{\partial\xi^2}+\frac{\partial^2\Phi}{\partial\zeta^2}+\frac{3}{R(1-\frac{\xi}{R})}\frac{\partial\Phi}{\partial\xi}+2c=0 \tag{5}$$

Using the new coordinates, the shearing-stress components given by Eq. (2) and the corresponding value of the torque given by Eq. (3) can be written as

$$\tau_{r\theta}=\frac{-G}{(1-\frac{\xi}{R})^2}\frac{\partial\Phi}{\partial\zeta},\qquad \tau_{\theta z}=\frac{-G}{(1-\frac{\xi}{R})^2}\frac{\partial\Phi}{\partial\xi} \tag{6}$$

$$M_t=\int\int(\tau_{r\theta}\zeta+\tau_{\theta z}(R-\xi))\,d\zeta d\xi \tag{7}$$

Using dimensionless distance coordinates X and Y, one can rewrite Eqs. (5), (6), and (7), respectively, as

$$\frac{\partial^2\Phi}{\partial X^2}+\left(\frac{a}{b}\right)^2\frac{\partial^2\Phi}{\partial Y^2}+\frac{3}{\frac{R}{2a}-(X-\frac{1}{2})}\frac{\partial\Phi}{\partial X}=-2c\,(2a)^2 \tag{8}$$

$$\tau_{r\theta}=\frac{-G}{(1-\frac{2aX}{R}+\frac{a}{R})^2}\frac{1}{2b}\frac{\partial\Phi}{\partial Y} \tag{9}$$

$$\tau_{\theta z}=\frac{-G}{(1-\frac{2aX}{R}+\frac{a}{R})^2}\frac{1}{2a}\frac{\partial\Phi}{\partial X} \tag{10}$$

$$M_t=(2a)(2b)\int\int(\tau_{r\theta}(2bY-b)+\tau_{\theta z}(R-2aX+a))\,dYdX \tag{11}$$

where $X=(\xi+a)/2a$ and $Y=(\zeta+b)/2b$.

Considering the boundary conditions, the resultant surface shear stress at the boundary must be in the direction of the tangent to the boundary; hence

$$\frac{GR^2}{r^2}\left(\frac{\partial\Phi}{\partial\zeta}\frac{d\zeta}{ds}+\frac{\partial\Phi}{\partial\xi}\frac{d\xi}{ds}\right)=0 \tag{12}$$

This shows that the stress function ϕ must be constant along the boundary of the cross section. In the case of singly connected boundaries, this constant can be chosen arbitrarily, and can be equal to zero.

3. Differential Quadrature Method

The Differential Quadrature Method(DQM) was introduced by Bellman and Casti.[5] By formulating the quadrature rule for a derivative as an analogous extension of quadrature for integrals in their introductory paper, they proposed the differential quadrature method as a new technique for the numerical solution of initial value problems of ordinary and partial differential equations. It was applied for the first time to static analysis of structural components by Jang, Bert and Striz.[6] The versatility of the DQM to engineering analysis in general and to structural analysis in particular is becoming increasingly evident by the related publications of recent years. Kang and Han[7] and Kang[8] applied the method to the analysis of a curved beam using classical and shear deformable beam theories and vibration analysis of curved beams. From a mathematical point of view, the application of the differential quadrature method to a partial differential equation can be expressed as follows:

$$L\{f(x)\}_i = \sum_{j=1}^{N} W_{ij}\, f(x_j) \text{ for } i,\ j=1,2,\ldots,N \quad (13)$$

where L denotes a differential operator, x_j are the discrete points considered in the domain, i are the row vectors of the N values, $f(x_j)$ are the function values at these points, W_{ij} are the weighting coefficients attached to these function values, and N denotes the number of discrete points in the domain. This equation, thus, can be expressed as the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

The general form of the function $f(x)$ is taken as

$$f_k(x) = x^{k-1} \text{ for } k = 1,2,3,\ldots,N \quad (14)$$

If the differential operator L represents an n^{th} derivative, then

$$\sum_{j=1}^{N} W_{ij} x_j^{k-1} = (k-1)(k-2)\cdots(k-n)x_i^{k-n-1} \text{ for } i,\ k = 1,2,\ldots,N \quad (15)$$

This expression represents N sets of N linear algebraic equations, giving a unique solution for the weighting coefficients, W_{ij}, since the coefficient matrix is a Vandermonde matrix which always has an inverse, as described by Hamming.[9]

4. Application

The differential quadrature approximations of the governing equation and the boundary conditions are given below.

Applying the DQM to the governing equation given by Eq. (8), gives

$$\sum_{k=1}^{N} B_{ik}\,\Phi_{kj} + \left(\frac{a}{b}\right)^2 \sum_{k=1}^{N} B_{jk}\,\Phi_{ik} + \frac{3}{\frac{R}{2a} - \left(X_i - \frac{1}{2}\right)} \sum_{k=1}^{N} A_{ik}\,\Phi_{kj} = -2c(2a)^2 \quad (16)$$

where A_{ik} and B_{ik} are the weighting coefficients for the first- and second-order derivatives, respectively, along the dimensionless axis.

Applying the DQM to the stress equations, given by Eqs. (9) and (10), gives

$$\tau_{r\theta} = \frac{-G}{\left(1 - \frac{2aX_i}{R} + \frac{a}{R}\right)^2} \frac{1}{2b} \sum_{k=1}^{N} A_{jk}\,\Phi_{ik} \quad (17)$$

$$\tau_{\theta z} = \frac{-G}{\left(1 - \frac{2aX_i}{R} + \frac{a}{R}\right)^2} \frac{1}{2a} \sum_{k=1}^{N} A_{ik}\,\Phi_{kj} \quad (18)$$

Applying the DQM to the torque equation given by Eqs. (11), gives

$$M_t = (2a)(2b) \sum_{i=1}^{N} \sum_{j=1}^{N} A_i A_j (\tau_{r\theta(ij)}(2bY_i - b) + \tau_{\theta z(ij)}(R - 2aX_i + a)) \quad (19)$$

where A_i are the weighting coefficients for the first-order integral along the dimensionless axis.

Applying the DQM to the boundary conditions gives

$$\Phi_{1j} = \Phi_{Nj} = 0 \quad \text{at} \quad X = 0,\ 1 \tag{20}$$

$$\Phi_{i1} = \Phi_{iN} = 0 \quad \text{at} \quad Y = 0,\ 1 \tag{21}$$

This set of equations together with the appropriate boundary conditions give a total of N×N number of simultaneous equations.

5. Numerical Results and Comparisons

The greatest shearing stress is calculated by the differential quadrature method. The third successive approximations for a square cross section with sides of length 2a given by Gohner[3] are compared with DQM. The third successive approximations for a rectangular cross section with sides of length 2a and thickness 2b given by Gohner[3] are also compared with DQM.

Table 1 presents the results of convergence studies relative to the number of grid points N with a/R = 1/20. The data show that the accuracy of the numerical solution increases with increasing N. Then, numerical instabilities arise if N becomes too large (possibly not greater than 19). In Table 2, the third approximations for the maximum stress at the inner point by Gohner[3] are compared with those by the DQM using thirteen grid points for the case of square cross section with sides of length 2a. From Table 2, it is seen that the difference between DQM and the third approximation solution in the shearing stress is increasing ratio by increasing a/R, and the stress is increasing by increasing sides of length. The values of circular and square cross are approximately 0.7448 and 0.6754 respectively when a/R is 1/10 (see Timoshenko and Goodier.[4] In Table 3, the solutions for the maximum stress at the inner point by Gohner[3] are compared with those by the DQM for the case of rectangular cross section with a/R = 1/4 and a/b = 4. All results are computed with thirteen grid points (see Kang).[8]

Table 1. The greatest shearing stress $\tau_{\theta z}\, a^3/PR$ for a range of grid points; square cross section with sides of length 2a

Gohner[3]	Number of grid points				
	7	9	11	13	15
0.6368	0.6388	0.6365	0.6366	0.6367	0.6367

Table 2. The greatest shearing stress $\tau_{\theta z}\, a^3/PR$ for a ratio of a/R; square cross section with sides of length 2a

a / R	Gohner[3]	D Q M
1 / 20	0.6368	0.6367
1 / 10	0.6754	0.6731
1 / 8	0.6953	0.6913
1 / 5	0.7574	0.7462
1 / 4	0.8010	0.7834
1 / 3	0.8773	0.8447

Table 3. The greatest shearing stress $\tau_{\theta z}\, b^3/PR$ for a ratio of a/R and a/b; rectangular cross section with sides of length 2a and 2b

a / R	a / b	Gohner[3]	D Q M
1 / 4	4	0.124	0.120

6. Conclusions

The differential quadrature method was used to compute the shearing stress of the rectangular cross section. The present method gives results which agree very well with the third successive approximations for the square cross section and rectangular cross section while requiring only a limited number of grid points.

References

1) A. M. Wahl, "Stresses in Heavy Closely Coiled Helical Springs," Tran. ASME, Vol. 51, pp. 185~200, 1929.
2) S. Timoshenko, Strength of Materials, Part II, D. Van Nostrand and Company, New York, 2nd ed., pp. 65~74, 1941.
3) O. Göhner, "Schubspannungsverteilung in Querschnitt einer Schraubenfeder," Ingenieur-Archiv, Vol. 1, pp. 619~644, 1930.
4) S. Timoshenko and J. N. Goodier, Theory of Elas-

ticity, McGraw-Hill, New York, 3rd ed., (International Student Edition) pp. 429~432, 1970, also see Timoshenko and Goodier, Theory of Elasticity, McGraw-Hill, New York, 2nd ed., pp. 391~395, 1951.
5) R. E. Bellman and J. Casti, "Differential Quadrature and Long-Term Integration," J. Math. Anal. Applic., Vol. 34, pp. 235~238, 1971.
6) S. K. Jang, C. B. Bert, and A. G. Striz, "Application of Differential Quadrature to Static Analysis of Structural Components," Int. J. Number. Mech. Engng, Vol. 28, pp. 561~577, 1989.
7) K. Kang and J. Han, "Analysis of a Curved beam Using Classical and Shear Deformable Beam Theories," Int. J. KSME., Vol. 12, pp. 244~256, 1998.
8) K. Kang, "Vibration Analysis of Curved Beams Using Differential Quadrature," J. KIIS., Vol. 14, pp. 199~207, 1999.
9) R. W. Hamming, Numerical Methods for Scientists and Engineers, McGraw-Hill, New York, 2nd ed., 1973.