International Journal of Reliability and Applications
Vol. 2, No. 4, pp. 263-268, 2001

Testing of Poisson Incidence Rate Restriction

Ramalingam Shanmugam
Department of Health Services Research
Southwest Texas State University, San Marcos, Texas 78666, USA

Karan Singh *
School of Public Health
University of Texas Health Sciences Center, FortWorth, TX76107, USA

Abstract. Shanmugam(1991) generalized the Poisson distribution to capture a restriction on the incidence rate θ (i.e. $\theta < \beta$, an unknown upper limit), and named it incidence rate restricted Poisson *(IRRP)* distribution. Using Neyman's C(α) concept, Shanmugam then devised a hypothesis testing procedure for β when θ remains unknown nuisance parameter. Shanmugam's C(α) based results involve inverse moments which are not easy tools. This article presents an alternate testing procedure based on likelihood ratio concept. It turns out that likelihood ratio test statistic offers more power than the C(α) test statistic. Numerical examples are included.

Key Words : *non-central chi-squared distribution.*

1. INTRODUCTION

Let X be a random variable denoting the number of chromatid aberrations in gene mutation analysis, the number of accidents per day in a location, or any count of rarity in a chance mechanism whose incidence rate is θ. In several situations like the ones described in this article, the incidence rate is likely to be, for some reasons, no more than an unknown amount $\beta = \frac{1}{\gamma}$. Shanmugam(1991) introduced an incidence rate restricted Poisson *(IRRP)* distribution to suit such a chance mechanism , and it is

$$f(x \mid \theta, \frac{1}{\gamma}) = \Pr[X = x] = (1 + \gamma x)^{x-1} (\theta e^{-\gamma\theta})^x / x! e^{\theta} \tag{1.1}$$

where $x = 0, 1, 2, ..,$ and $0 < \theta \leq \frac{1}{\gamma}$. Notice that when the restriction parameter $\beta = \frac{1}{\gamma} = \infty$, the distribution in (1.1) reduces to the usual Poisson distribution

$$\Pr[X = x] = e^{-\theta} \theta^x / x!. \tag{1.2}$$

*Supported by NSF grant # 123456
E-mail address : ksingh@hsc.unt.edu

Using C(α) concept, Shanmugam then devised a testing procedure for hypothesis about $\frac{1}{\gamma}$. The derivation of Shanmugam's C(α) results involve inverse moments which are not easy to deal with. Hence, this article presents an alternate and yet powerful (likelihood ratio) test statistic for testing hypothesis about $\frac{1}{\gamma}$. The power of the likelihood and C(α) tests are compared and illustrated.

2. INFERENCE PROCEDURE FOR RESTRICTION PARAMETER

Consider a random sample $x_{(n)} = (x_1, x_2, .., x_n)$ of observations on the number of tram accidents as stated in Table 1. In real life, the factors like the training of drivers and the experience gained by the agencies would necessarily impose a restriction on the accident rate θ. Consequently, the *IRRP* distribution is a natural choice for such data. When the influence of such factors is absent, the incidence rate θ is unrestricted and hence, $\frac{1}{\gamma} = \infty$ implying the usual Poisson distribution in (1.2) is appropriate. So, the absence versus presence of restriction on the incidence rate can be expressed in statistical framework as null $H_o : \frac{1}{\gamma} = \infty$ versus alternative $H_1 : 0 < \frac{1}{\gamma} < \infty$ hypothesis The likelihood ratio procedure is selected here to test H_o. The log-likelihood function with the *IRRP* distribution in (1.1) is

$$\ln L(\theta, \frac{1}{\gamma}) = n([\ln\theta - \gamma\theta]\overline{x} - \theta) + \sum[(x_i - 1)\ln(1 + \gamma x_i) - \ln x_i!]. \tag{2.1}$$

The maximum likelihood estimators *(MLE)* $\widehat{\theta}$ and $\widehat{\gamma}$ are obtained by iteratively solving the equations

$$\sum \frac{x_i(x_i - 1)}{\theta(\overline{x} - x_i) + x_i\overline{x}} = n \tag{2.2}$$

and

$$\gamma = \frac{1}{\theta} - \frac{1}{\overline{x}}. \tag{2.3}$$

Mimicing the arguments in Consul and Shoukri(1984), it can be shown that the solutions of (2.3) and (2.2) are unique and admissible. For initial value, moment estimator *(ME)*

$$\widetilde{\theta} = \frac{\overline{x}^{\frac{3}{2}}}{s_x}$$

can be adapted, where $\overline{x}$ and s_x denote the sample mean and standard deviation respectively. When the null hypothesis is true, (i.e. $\frac{1}{\gamma} = \infty$), and hence, the MLE of θ is simply $\theta^* = \overline{x}$. With these MLEs, the likelihood ratio becomes

$$\Lambda(x_{(n)}) = \frac{L(\theta^*, \frac{1}{\gamma} = \infty)}{L(\widehat{\theta}, \widehat{\gamma})}.$$

The critical region is then of the form $\Lambda(x_{(n)}) \leq C$ where C is a suitable constant chosen to meet the required size of the test. The likelihood ratio under the null hypothesis $H_o : \gamma = \frac{1}{\beta} = 0$, is therefore

$$\Lambda(x_{(n)}) = (\frac{\widehat{x}}{\overline{\theta}})^{n\widehat{x}} e^{\widehat{\theta}\widehat{\gamma} n\overline{x} + n(\widehat{\theta} - \overline{x})} / \Pi_{i=1}^{i=n} (1 + \widehat{\gamma} x_i)^{x_i - 1}.$$

Then,

$$\begin{aligned} T_\gamma &= -\ln \Lambda(x_{(n)}) \\ &= -2[\widehat{\theta}\widehat{\gamma} n\overline{x} + n(\widehat{\theta} - \overline{x}) - n(\ln \frac{\widehat{\theta}}{\overline{x}})\overline{x} - \sum_{i=1}^{i=n}(x_i - 1)\ln(1 + \widehat{\gamma} x_i). \qquad (2.4) \end{aligned}$$

Note that (according to Wald(1943))that T_γ asymptotically follows a non-central chi-squared distribution with one degree of freedom *(df)* with a non-central parameter

$$\delta_\gamma = (\widehat{\gamma} - \gamma_o)^2 / Var(\widehat{\gamma}) \qquad (2.5)$$

where $\frac{1}{Var(\widehat{\gamma})} = -E[\frac{\partial^2 \ln L}{\partial \gamma^2}]$. When H_o is true, $\delta_\gamma = 0$ and the T follows a central chi-squared distribution. This means that the hypothesis $H_o : \frac{1}{\gamma} = \infty$ should be rejected when the likelihood ratio statistic $T_\gamma > \chi^2_{1df,\alpha}$, the critical value based on the 100(1-α)th percentile from chi-squared distribution with 1 df and a significance level $\alpha \in (0, 1)$.

2.1 Power of the Test

Under a finite value of $\beta = \frac{1}{\gamma}$, the distribution of T_γ follows a non-central chi-squared distribution as described earlier, and this enables the computation of the asymptotic power. To evaluate such power, the asymptotic variance-covariance matrix is needed and it is the inverse of the information matrix

$$\begin{aligned} I &= \begin{bmatrix} E(-\frac{\partial^2 \ln f(x|\theta,\frac{1}{\gamma})}{\partial\theta^2}) & E(-\frac{\partial^2 \ln f(x|\theta,\frac{1}{\gamma})}{\partial\theta\partial\gamma}) \\ E(-\frac{\partial^2 \ln f(x|\theta,\frac{1}{\gamma})}{\partial\theta\partial\gamma}) & E(-\frac{\partial^2 \ln f(x|\theta,\frac{1}{\gamma})}{\partial\gamma^2}) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\theta^2}E(X) & E(X) \\ E(X) & E(\frac{X^2(X-1)}{(1+X\gamma)^2}) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\theta(1-\theta\gamma)} & \frac{\theta}{(1-\theta\gamma)} \\ \frac{\theta}{(1-\theta\gamma)} & \frac{\theta^3}{(1-\theta\gamma)} + \frac{2\theta^3}{(1+2\theta\gamma)} \end{bmatrix}. \end{aligned}$$

The asymptotic variance of the MLE of $\widehat{\gamma}$ is

$$Var(\widehat{\gamma}) = \frac{1 + 2\gamma}{2n\theta^2}. \qquad (2.6)$$

We now return to discuss the situation under alternative hypothesis $H_1 : \gamma = \gamma_1$. The likelihood ratio statistic T_γ follows a non-central chi-squared distribution with one df with a non-central parameter

$$\delta_\gamma = \gamma_1^2(\frac{2n\widehat{\theta_1^2}}{1 + 2\gamma_1}) \qquad (2.7)$$

where $\hat{\theta_1}$ is the MLE of θ under $H_1 : \gamma = \gamma_1$. Using a result in Kendall and Stuart(1979, page 245), the non-central chi-squared distribution $\chi^2_{1df}(\delta)$ can be approximated by a central chi-squared distribution $\chi^2_{\delta^* df}$ as it follows.

$$\chi^2_{1df}(\delta) \approx^d \rho\chi^2_{\delta^* df}$$

where the symbol $\approx^d$ stands for approximately distributed,

$$\rho = 1 + \frac{\delta}{1+\delta} \tag{2.8}$$

and

$$\delta^* = 1 + \frac{\delta^2}{1+2\delta}. \tag{2.9}$$

The power of the likelihood ratio test is therefore

$$\begin{aligned} \Phi_\gamma &= power = \Pr[T_\gamma > \chi^2_{1df,\alpha}(\delta_\gamma)] \\ &= \Pr[\chi^2_{\delta^*_\gamma df} > \frac{1}{\rho_\gamma}\chi^2_{1df,\alpha}]. \end{aligned} \tag{2.10}$$

2.2 Illustration

Consider Leiter and Hamdan's data, as reported in Shanmugam(1991), about the number of accidents that occurred on Interstate 95 in Virginia State during 639 days. The estimates are found to be $\hat{\theta} = 0.805$ and $\hat{\beta} = 12.3$. Using the test criterion in (2.4), the p-value is found to be 0.02 which suggests the plausibility of IRRP distribution in (1.1) for the data.

Substituting $\beta = 8.36$ in (2.10) which is a value Shanmugam used for the Neyman C(α) procedure, the likelihood ratio test statistic in (2.10) yields a power of $\Phi_\gamma = 0.885$. This is a notable improvement from the power 0.587 of Neyman C(α) test in Shanmugam(1991)

For another example, consider the distribution of 102 spiders among 240 pieces of cover cited in Janardan, Kerster, and Schaeffer(1979). Assuming that an IRRP distribution fits well the data, the MLEs are found to be $\hat{\theta} = 0.411$ and $\hat{\beta} = 12.856$. The p-value for rejecting $H_o : \beta = \infty$ is noticed to be 0.487, not a significant amount. This confirms Janardan's conclusion that the spider's behavior is well described by the usual Poisson distribution. In this context, it is worth pointing out that over parametrization does not necessarily guarantee the success in fitting the data.

3. TESTING THE INCIDENCE PARAMETER

After establishing the plausibility of IRRP distribution in (1.1) for a given data based on either the likelihood ratio or Neyman C(α) test for β, it is appropriate to devise an analogous test for the incidence parameter θ as well. To test the null hypothesis $H_o : \theta = \theta_o$ against an alternative hypothesis $H_a : \theta \neq \theta_1$, the steps in Section 2 need be repeated by interchanging the role of θ and β. The likelihood ratio test statistic becomes

$$T_\theta = -2[\sum x_i[\ln(\frac{\theta_o}{\widehat{\theta}}) + \widehat{\gamma}\widehat{\theta} - \widehat{\gamma}_{\theta_o}\widehat{\theta}] + n(\widehat{\theta} - \theta_o) + \sum_{i=1}^{i=n}(x_i - 1)[\ln(\frac{1 + \widehat{\gamma}_{\theta_o}x_i}{1 + \widehat{\gamma}x_i}) \quad (3.1)$$

where γ_{θ_o} is the MLE of γ when $\theta = \theta_o$ is true. The null hypothesis $H_o : \theta = \theta_o$ should be rejected when $T_\theta > \chi^2_{1df,\alpha}$. When the alternative hypothesis $H_a : \theta = \theta_1$ is true, the test statistic T_θ follows a non-central chi-squared distribution with one degree of freedom with the non-centrality parameter

$$\begin{aligned} \delta_\theta &= (\theta_1 - \theta_o)^2/Var(\widehat{\theta}) \\ &= 2n(\theta_1 - \theta_o)^2/\theta_1(\theta_1 + 2). \end{aligned} \quad (3.2)$$

The non-central chi-squared distribution of T_θ can be approximated by a central chi-squared distribution with a co-factor

$$\rho_\theta = 1 + \frac{2n(\theta_1 - \theta_o)^2/\theta_1(\theta_1 + 2)}{1 + 2n(\theta_1 - \theta_o)^2/\theta_1(\theta_1 + 2)} \quad (3.3)$$

and the degrees of freedom

$$\delta^*_\theta = 1 + \frac{[n(\theta_1 - \theta_o)^2/\theta_1(\theta_1 + 2)]^2}{\frac{1}{4} + n(\theta_1 - \theta_o)^2/\theta_1(\theta_1 + 2)}. \quad (3.4)$$

The power is computed to be

$$\begin{aligned} \Phi_\theta &= \Pr[T_\theta > \chi^2_{1df,\alpha}(\delta_\theta)] \\ &\approx \Pr[\chi^2_{\delta^*_\theta df} > \frac{1}{\rho_\theta}\chi^2_{1df,\alpha}]. \end{aligned} \quad (3.5)$$

3.1 Illustration

To illustrate the likelihood ratio test on the incidence parameter, consider the data on accidents which occurred by $n = 134$ tram drivers in Belgrade between 1965 and 1970 as reported in Milosevic and Vucinic(1974) (see Table 1). Note that IRRP distribution fits the data better (with $\chi^2_{1df,0.66} = 6.76$ and $\alpha = 0.66$) than the Poisson, negative binomial, or Short distribution. The estimates are $\widehat{\theta} = 3.724$ and $\widehat{\beta} = 9.845$

Suppose the null hypothesis $H_o : \theta = \theta_o = 3.72$ is to be tested against an alternative $H_a : \theta = \theta_1 = 2.0$. The power of the likelihood test statistic for rejecting $H_o : \theta = \theta_o = 3.72$ would depend on the non-centrality parameter value $\delta_\theta = 69.305$,the co-factor $\rho_\theta = 1.986$, and the approximate degrees of freedom $\delta^*_\theta = 35$ according to (3.2), (3.3), and (3.4). The power of the likelihood ratio test is

$$\Phi_\theta = \Pr[\chi^2_{\delta^*_\theta df} > \frac{1}{\rho_\theta}\chi^2_{1df,\alpha}] = \Pr[\chi^2_{35df} > 3.404] = 0.998.$$

Table 1. Accidents made by 134 tram drivers over the period 1965-1970

$\sqrt{\#accidents}$	#drivers	Poisson fit	Neg.Binomial fit	Short fit	IRRP fit
0	1	0.33	3.95	2.40	3.23
1	8	2.00	8.86	8.14	8.25
2	14	6.01	12.72	14.09	12.66
3	17	12.00	14.82	16.93	15.24
4	16	17.98	15.27	16.41	15.87
5	19	21.55	14.50	14.30	15.06
6	16	21.52	12.98	12.10	13.37
7	9	18.42	11.12	10.26	11.32
8	6	13.80	9.21	8.66	9.25
9	6	9.19	7.41	7.17	7.35
10	3	5.51	5.84	5.76	5.72
11	4	3.00	4.51	4.52	4.37
12	14	2.68	12.22	12.28	11.69
χ^2		72.51	10.76	9.89	6.76
df		8	9	8	9
p-value		<0.0001	0.29	0.27	0.66

REFERENCES

Consul, P. C. and Shoukri, M. M. (1984). Maximum likelihood estimation for the generalized Poisson distribution, *Communications in Statistics*, **A13(12)**, 1533-1547.

Janardan, K. G. , Kerster, H. W., and Schaeffer, D. J. (1979). Biological applications of the Lagrangian Poisson distribution, *Bioscience*, **29**, 599-602.

Kendall, M. G. and Stuart, A. (1979). *The Advanced theory of Statistics, Vol. 2*, Charles Griffin Press, London.

Milosevic, S. and Vucinic, S. (1975). Statistical study of Tram driver accidents, *Accident Analysis and Prevention*, **7**, 1-7.

Shanmugam, R. (1991). Incidence Rate Restricted Poissonness, *Sankhya*, **53**, 191-201.

Wald, A. (1943). Tests of statistical hypotheses concerning several parameters when the number of observations is large, *Transactions of American Mathematical Society*, **54**, 426.

Wilks, S. S. (1938). The large sample distribution of the likelihood ratio for testing composite hypotheses, *Annals of Mathematical Statistics*, **9**, 60.