International Journal of Reliability and Applications
Vol. 2, No. 4, pp. 241-251, 2001

Optimal Burn-in Time under Cumulative Pro-Rata Replacement Warranty

Won Young Yun*
Department of Industrial Engineering
Pusan National University, Pusan, 609-735, Korea

Yang Woo Lee
Department of Industrial Engineering
Pusan National University, Pusan, 609-735, Korea

Il Han Chung
Department of Industrial Engineering
Pusan National University, Pusan, 609-735, Korea

Luis Ferreira
School of Civil Engineering
Queensland University of Technology, Brisbane, Australia

Abstract. In this paper, optimal burn-in time to minimize the total mean cost, which is the sum of manufacturing cost with burn-in and cumulative warranty-related cost, is obtained. When the products with cumulative pro-rata warranty have high failure rate in the early period (infant mortality period), a burn-in procedure is adopted to eliminate early product failures. After burn-in, the posterior product life distribution and the warranty-related cost are dependent on burn-in time; long burn-in period may reduce the warranty-related cost, but it increases the manufacturing cost. The paper provides a methodology to obtain total mean cost under burn-in and cumulative pro-rata warranty. Property of the optimal burn-in time is analyzed, and numerical examples and sensitivity analysis are studied.

Key Words : *burn-in, cumulative pro-rata warranty, total mean cost, failure rate, life distribution.*

1. INTRODUCTION

Most durable products are sold today with some type of warranty to protect consumers from unexpected early product failures. The manufacturer's warranty-related costs are becoming a significant portion of production cost. High product reliability in the early period of the product life cycle is crucial to reduce such warranty-related costs.

* *E-mail address : wonyun@hyowon.pusan.ac.kr*

There are generally three phases in the product life cycle in terms of failure rate. These phases can be represented by the bathtub pattern as Figure 1. The early stage, with decreasing failure (hazard) rate, is called the infant mortality period; the second stage, with a constant failure rate, is called the normal (useful) period; and the last stage, with an increasing failure rate, is called the wear-out period.

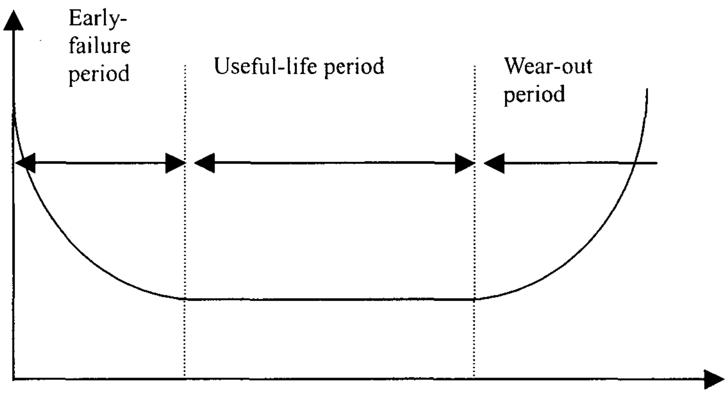

Figure 1. A typical bathtub curve

To reduce damage from early failures, a burn-in procedure is carried out by operating the products under electrical or thermal conditions that approximate the working conditions in field operation before shipment to customers. Early failures result in a high warranty cost, which is dependent on burn-in period and warranty type. We consider a special type of warranty, cumulative pro-rata replacement warranty (cumulative PRW) which differs significantly from standard warranty policies. The cumulative warranty covers the group of items for a total service time of nT, rather than covering each item separately for a period T. Cumulative warranty policies have been proposed for use in the United States in military acquisition.

This paper deals with the economic optimization problem of how long burn-in procedure should be to minimize the total mean cost, which is the sum of manufacturing cost with burn-in and warranty cost under cumulative warranties.

The studies for burn-in test began with the advent of transistors in the early 1950's (Kececioglu and Sun (1997), Kuo and Kuo (1983), Leemis and Beneke (1990), Block and Savits (1997)).

Nguyen and Murthy (1982) first proposed a model to determine the optimal burn-in time for products sold with warranty. They considered two types of warranty policy (failure-free and rebate policies) and derived the total cost as the sum of the manufacturing and the warranty cost, for both repairable and nonrepairable products. Mi (1997) considered the same situation as Nguyen and Murthy (1982) with assumption that burn-in procedure continued until the first product surviving the burn-in period, although they used a different cost structure. Mi (1997) also proved some important properties that the optimal burn-in time occurs no later than the first change point of bathtub failure rate. Mi (1999) compared policies with renewable warranties and burn-in. Recently, Monga and Zuo (1998) studied the reliability-based design of a series-parallel system considering burn-in, warranty and maintenance. They assumed that the system life cycle cost included costs of burn-in, warranty, installation, preventive maintenance, and minimal repair. They obtained optimal values of system design, burn-in time, preventive maintenance intervals, and replacement time. Kar and Nachlas (1997) studied warranty and burn-in strategies together, in order to examine the possible benefits of coordinated strategies for product performance management.

In this paper, a cumulative PRW can be considered as follows;

Cumulative PRW: A lot of n items is purchased at cost nc_s and warranted for a total period nT. The n items may be used either individually or in batches. The total service time S_n is calculated after failure of the last item in the lot. If $S_n < nT$, the buyer is given a refund in the amount of $c_s(n - S_n / T)$.

Burn-in Procedure(see Mi(1997)): We consider a nonrepairable product. For given burn-in period b, all the new product is tested under an environment similar to field condition. Any failed product during burn-in period is replaced with a new one. Burn-in test is continued until the first product surviving the burn-in period b is obtained.

In addition, we assume that the products have a bathtub failure pattern. Given cumulative PRW and bathtub failure rate, we consider optimal burn-in problem.

2. TOTAL MEAN COST MODEL

In this section, we derive cost elements under burn-in and cumulative PRW and the property of optimal burn-in period is proved.

Notation

$F(t)$	:	cumulative distribution function of failure time
$f(t)$	:	failure time probability density function.
$\bar{F}(t)$	:	survival function = $1 - F(t)$.
$r(t)$	:	hazard rate function $= f(t)/\bar{F}(t)$.
b	:	burn-in time.
$F_b(t)$	:	distribution function of product survived burn-in period b $=[F(b+t)-F(t)]/\bar{F}(b)$.
T	:	warranty period on individual item.
$M(t)$	:	renewal function with interarrival time distribution $F(t)$.
$M_b(t)$	:	renewal function with interarrival time distribution $F_b(t)$.
$M_{CF}(t)$	:	expected number of free-replacement under cumulative warranty.
$M_{CF,b}(t)$	:	expected number of free-replacement under cumulative warranty, when burn-in procedure is used.
$R(t)$	:	rebate function.

c_0	:	manufacturing cost per item without burn-in.
c_1	:	fixed set-up cost of burn-in per item.
c_2	:	cost of burn-in per item per unit time.
c_3	:	shop replacement cost per failure dering burn-in period, includes the manufacturing cost and set-up cost.
c_s	:	cost occurred to rebate the product when a failure occurs at $t = 0$ under cumulative PRW.
$h(b)$	:	manufacturing cost incurred until the first item survives the burn-in time.
$v(b)$	:	expected manufacturing cost $= E[h(b)]$.
$k_{CP}(n,b)$	:	Warranty cost for a lot size n, and burn-in period b.
$w_{CP}(n,b)$	:	Expected warranty cost $= E[k_{CP}(n,b)]$.
$C_{CP}(T, n)$	:	total mean cost for warranty period T, lot size n, and no burn-in.
$C_{CP}(T, n, b)$	:	total mean cost for warranty period T, lot size n, and burn-in period b.
$CU_{CP}(T, n, b)$	:	per-item total mean cost for warranty period T, lot size n, and burn-in period b.

2.1 Mean Burn-in Cost

Under given burn-in, let Z be the total burn-in time until first product survives the burn-in time b, and $X_1, \ldots, Xn, \ldots$ be independently and identically distributed lifetimes of the product with common c.d.f. F. Let also $\eta - 1$ be the random variable that is the number of shop replacements until the first product surviving burn-in time is obtained. The manufacturing cost incurred until the first product survives the burn-in time b is given by

$$h(b) = c_0 + c_1 + c_2 Z + c_3(\eta - 1), \tag{2.1}$$

where $Z = \sum_{i=1}^{\eta-1} X_i + b$.

$$E[\eta - 1] = \frac{[1 - \overline{F}(b)]}{\overline{F}(b)} = \frac{F(b)}{\overline{F}(b)}. \tag{2.2}$$

Product lifetime $X_1, X_2, \ldots, X_\eta$ has the following distribution:

$$\Pr\{X_i < x\} = \begin{cases} 1 & x \geq b \\ \dfrac{F(x)}{F(b)} & 0 \leq x \leq b \quad \text{for } i = 1, 2, \ldots, \eta - 1. \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

Thus

$$E[X_i] = \int_0^b (1 - \Pr\{X_i < x\})dx$$
$$= \left(\frac{1}{F(b)}\right)\left[bF(b) - \int_0^b F(x)dx\right] \quad \text{for } i = 1, 2, \ldots, \eta - 1. \tag{2.4}$$

and

$$E[Z] = \left(\frac{F(b)}{\bar{F}(b)}\right)\left(\frac{1}{F(b)}\right)\left[bF(b) - \int_0^b F(x)dx\right] + b$$
$$= \frac{\int_0^b [1 - F(x)]dx}{\bar{F}(b)} \tag{2.5}$$

Therefore, the mean burn-in cost, $v(b)$ is given by

$$v(b) = E[h(b)] = c_0 + c_1 + c_2 \frac{\int_0^b \bar{F}(t)dt}{\bar{F}(b)} + c_3 \frac{F(b)}{\bar{F}(b)}. \tag{2.6}$$

2.2 Mean Cumulative Warranty Cost

For a product survived burn-in time b, its distribution function is

$$F_b(t) = \frac{F(b+t) - F(b)}{\bar{F}(b)}, \quad t \geq 0. \tag{2.7}$$

To acquire warranty cost for Cumulative PRW, let $R(t)$ be a rebate function, which means refund amount to buyer when nth failure occurs before cumulative warranty period.

Generally, $R(t)$ is decreasing in $t \in [0, nT]$ and zero outside this interval. The warranty cost is the refund amount under Cumulative PRW. It depends on nth failure time, S_n. Thus warranty cost for Cumulative PRW is given by

$$k_{CP}(T,n) = \begin{cases} R(S_n) & \text{for } S_n \leq nT \\ 0 & \text{otherwise,} \end{cases} \tag{2.8}$$

where $S_n = \sum_{i=1}^{n} X_i$ and X_i has common cdf. F_b.

Thus, the expected warranty cost for Cumulative PRW is given by

$$w_{CP}(T,n,b) = \int_0^{nT} R(t)dF^{(n)}{}_b(t). \tag{2.9}$$

In this paper, a specific function is used as follows

$$R(t) = \begin{cases} (nT - t)/nT \times (n \times c_s) & \text{for } t \leq nT \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

2.3 Total Mean Cost

Under burn-in period, b and cumulative PRW, the total mean cost is a sum of warranty and burn-in costs given by

$$C_{CP}(T, n, b) = nv(b) + w_{CP,b}(T, n)$$
$$= nv(b) + \int_0^{nT} R(t)dF_b^{(n)}(t), \quad (2.11)$$

and the total mean cost per-item is given by

$$CU_{CP}(T, n, b) = v(b) + \int_0^{nT} R(t)dF_b^{(n)}(t) \;/\; n. \quad (2.12)$$

Let's assume that the product has bathtub failure rate, $r(t)$ if there exist t_1 and t_2 such that:

$$r(t) \text{ is } \begin{cases} \text{strictly decreasing,} & \text{for } 0 \le t \le t_1, \\ \text{constant,} & \text{for } t_1 \le t \le t_2, \\ \text{strictly increasing,} & \text{for } t_2 \le t, \end{cases}$$

where t_1 and t_2 are called the change points of $r(t)$. The time interval $[0, t_1]$ is called the infant mortality period; the interval $[t_1, t_2]$ where $r(t)$ is flat and attains its minimum value is called the normal operating period or useful period; the interval $[t_2, \infty)$ is called the wear-out period.

Then, for a Cumulative PRW policy with burn-in procedure, optimal burn-in time b^* to minimize the total mean cost function, Equation (2.11) occurs no later than first change point t_1 (Refer Mi(1997)).

3. NUMERICAL EXAMPLES

In this section, we consider some distribution functions and optimal burn-in times for each one.

Weibull-Exponential Distribution

Chou, and Tang (1992) introduced the following distribution to model the failure pattern of product in the burn-in studies. The failure rate function is given by

$$r(t) = \begin{cases} \beta(1/\alpha)^{\beta} t^{\beta-1} & 0 \le t \le t_1 \\ \beta(1/\alpha)^{\beta} t_1^{\beta-1} & t_1 \le t, \end{cases} \quad (3.1)$$

where $0 < \beta < 1$ is the shape parameter, α is the scale parameter, and t_1 is the change-point. Because $0 < \beta < 1$, the failure rate function is strictly decreasing in the interval $[0, t_1]$ and stays at a constant level $\beta(1/\alpha)^{\beta} t_1^{\beta-1}$, for $t_1 \le t$.

Figure 2 shows the relation of the optimal burn-in time and the lot size when $c_0 = 126$, $c_1 = 1$, $c_2 = 0.01$, $c_3 = 127$, $c_s = 400$, $\alpha=12$, $\beta=0.55$, $\alpha_1=12$, $\beta_1= 1/0.55$, and $t_1= 2$, and the change of the optimal burn-in time according to β when $c_0 = 126$, $c_1 = 1$, $c_2 = 0.01$, $c_3 = 127$, $c_s = 400$, $\alpha=12$, and $t_1= 2$.

The underlying distribution function F and model parameters results in different optimal burn-in period. Now, the effects of the lot size n, and the individual warranty period T on the per-item total mean cost and the optimal burn-in time are studied. Then the effects of the model parameter c_i on the optimal burn-in time are also studied for Weibull-Exponential distribution with $\alpha=12$, $\beta=0.55$, and $t_1=2$ is considered. Its distribution function is as follows:

$$F(t)=\begin{cases}1-\exp[-(t/\alpha)^{\beta}] & \text{for } 0\le t\le t_1\\ 1-\exp[-(t_1/\alpha)^{\beta}-\beta(1/\alpha)^{\beta}t_1^{\alpha-1}(t-t_1)] & \text{for } t_1\le t,\end{cases} \quad (3.2)$$

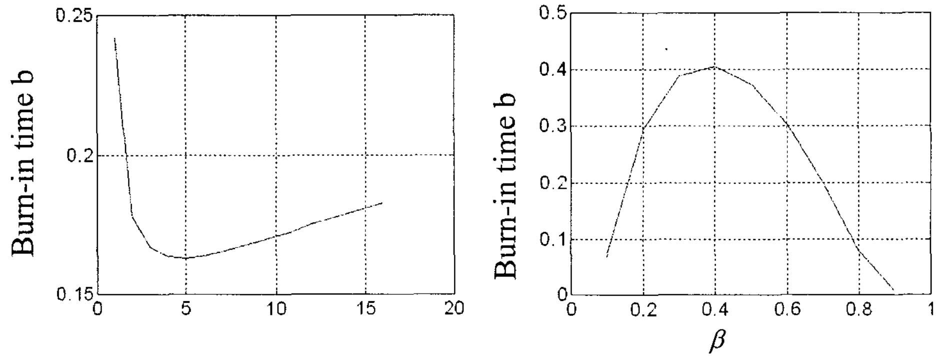

Figure 2. Optimal burn-in time for Weibull-Exponential with $t_1=2$ when c_0 = 126, c_1 = 1, c_2 = 0.01, c_3 = 127, and c_s = 400.

Table 1 shows the effects of the lot size n and the individual warranty period T on the per-item total mean cost and the burn-in time respectively for Cumulative PRW. As shown in Table 1, when the lot size n increases, the optimal burn-in time b^* decreases. And when the individual warranty period T increases, the optimal burn-in time b^* increases and then decreases. The change of the lot size n is more sensitive to the change of the optimal burn-in time b^* than that of the individual warranty period T.

Table 1. Optimal burn-in time, and per-item total mean cost for Weibull-Exponential with $\alpha=12, \beta=0.55$, and $t_1=2$ when c_0 = 100, c_1 = 1, c_2 = 0.01, c_3 = 102, and c_s = 300

		$n=1$		$n=2$		$N=3$		$n=4$	
		CU_{PR}	b^*	CU_{PR}	b^*	CU_{PR}	b^*	CU_{PR}	b^*
T	1	140.31	0.1640	112.81	0.0000	104.68	0.0000	102.30	0.0000
	2	157.60	0.2572	123.76	0.0140	111.01	0.0000	105.78	0.0000
	4	183.23	0.2837	145.49	0.0797	129.18	0.0118	120.21	0.0000
	6	204.48	0.2447	168.16	0.1289	152.50	0.0821	143.43	0.0575
	8	222.78	0.1938	192.17	0.1409	179.52	0.1328	172.09	0.1310
	10	238.99	0.1356	216.06	0.1226	207.30	0.1320	202.52	0.1414
	12	253.92	0.0832	238.15	0.0885	232.93	0.0995	230.49	0.1079

Table 2. Optimal burn-in time for Weibull-Exponential with $\alpha = 12, \beta = 0.55,$ and $t_1 = 2$.

T=8, t_l=2, C_0=100				C_3/C_0			
				1	1.02	1.05	1.1
Cs / Co	1.5	C_2 / C_0	0	0.000048	0.000048	0.000048	0.000048
			0.0001	0.000048	0.000048	0.000048	0.000048
			0.0002	0.000048	0.000048	0.000048	0.000048
			0.0005	0.000048	0.000048	0.000048	0.000048
	2.0	C_2 / C_0	0	0.006621	0.004299	0.001828	0.000048
			0.0001	0.006611	0.004292	0.001825	0.000048
			0.0002	0.006602	0.004286	0.001823	0.000048
			0.0005	0.006574	0.004268	0.001815	0.000048
	3.0	C_2 / C_0	0	0.158823	0.147308	0.131103	0.106793
			0.0001	0.158647	0.147145	0.13096	0.106678
			0.0002	0.158471	0.146983	0.130818	0.106564
			0.0005	0.157946	0.146498	0.130392	0.106222
	4.0	C_2 / C_0	0	0.358652	0.343325	0.321219	0.286652
			0.0001	0.358324	0.343013	0.320929	0.286397
			0.0002	0.357996	0.342701	0.32064	0.286142
			0.0005	0.357016	0.341768	0.319775	0.285381

Table 2 shows the effects of the cost parameter c_i. If the cost of burn-in per item per unit time c_2 increases, the optimal burn-in time b^* decreases. When the shop replacement cost c_3 increases, the optimal burn-in time b^* decreases. And when the rebate cost c_s increases, the optimal burn-in time b^* increases.

Weibull-Exponential-Weibull Distribution

We use the following distribution to model the bathtub failure pattern. The failure rate function is given by

$$r(t)=\begin{cases}\beta_1(1/\alpha_1)^{\beta_1}t^{\beta_1-1} & 0\le t\le t_1\\ \beta_1(1/\alpha_1)^{\beta_1}t_1^{\beta_1-1} & t_1\le t<t_2\\ \beta_1(1/\alpha_1)^{\beta_1}t_1^{\beta_1-1}+\beta_2(1/\alpha_2)^{\beta_2}(t-t_2)^{\beta_2-1} & t_2\le t\end{cases} \quad (3.3)$$

where $0<\beta_1<1$ and $\beta_2>1$ are the shape parameters, α is the scale parameter, and t_1 and t_2 are the change-points.

Figure 3 shows the relation of the optimal burn-in time and the lot size when c_0 = 126, c_1 = 1, c_2 = 0.01, c_3 = 127, c_s = 400, α_1=12, β_1=0.55, α_2=12, β_2= 1/0.55, t_1= 2, and t_2= 8, and the change of the optimal burn-in time according to β when c_0 = 126, c_1 = 1, c_2 = 0.01, c_3 = 127, c_s = 400, α_1=12, α_2=12, β_2= 1/0.55, t_1= 2, and t_2= 8.

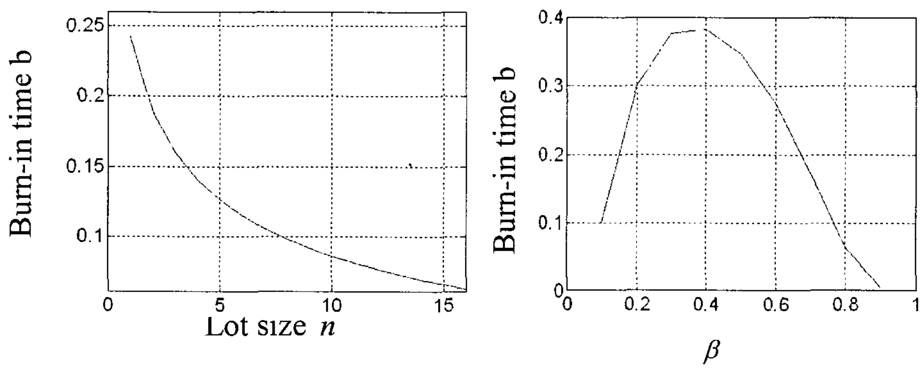

Figure 3. Optimal burn-in time for Weibull-Exponential-Weibull with $t_1=2$, and $t_2=8$ when c_0 = 126, c_1 = 1, c_2 = 0.01, c_3 = 127, and c_s = 400.

Table 3. Optimal burn-in time, and per-item total mean cost for Weibull-Exponential-Weibull with $\alpha_1=12$,$\beta_1=0.55$, $\alpha_2=12$, $\beta_2=1/0.55$, $t_1=2$, and $t_2=8$ when c_0 = 100, c_1 = 1, c_2 = 0.01, c_3 = 102, and c_s = 300.

		$n=1$		$n=2$		$n=3$		$n=4$	
		CU_{PR}	b^*	CU_{PR}	b^*	CU_{PR}	b^*	CU_{PR}	b^*
T	2	140.31	0.1640	112.81	0.0000	104.68	0.0000	102.30	0.0000
	3	157.60	0.2572	123.76	0.0140	111.01	0.0000	105.78	0.0000
	4	183.23	0.2837	145.49	0.0797	129.03	0.0109	119.73	0.0000
	6	204.48	0.2447	167.64	0.1301	150.43	0.0715	140.12	0.0381
	8	222.78	0.1940	189.27	0.1471	173.16	0.1227	163.27	0.1067
	10	238.65	0.1440	209.53	0.1386	195.67	0.1410	187.21	0.1453
	12	252.47	0.0991	227.95	0.1150	216.75	0.1316	210.17	0.1467

Table 3 shows the effects of the lot size n and the individual warranty period T on the per-item total mean cost and the burn-in time respectively for Cumulative PRW As shown in Table 3, when the lot size n increases, the optimal burn-in time b^* decreases. And when the individual warranty period T increases, the optimal burn-in time b^* increases and then decreases. The change of the lot size n is more sensitive to the change of the optimal burn-in time b^* than that of the individual warranty period T.

Table 4 shows the effects of the cost parameter c_i. If the cost of burn-in per item per unit time c_2 increases, the optimal burn-in time b^* decreases. When the shop replacement cost c_3 increases, the optimal burn-in time b^* decreases. And when the rebate cost c_s increases, the optimal burn-in time b^* increases.

Table 4. Optimal burn-in time for Weibull-Exponential-Weibull with $\alpha_1 = 12$, $\beta_1 = 0.55$, $\alpha_2 = 12$, $\beta_2 = 1/0.55$

T=8, t_1=2, t_2=4, C_0=100				C_3/C_0			
				1	1.02	1.05	1.1
Cs/C_0	1.5	C_2/C_0	0	0.000048	0.000048	0.000048	0.000048
			0.0001	0.000048	0.000048	0.000048	0.000048
			0.0002	0.000048	0.000048	0.000048	0.000048
			0.0005	0.000048	0.000048	0.000048	0.000048
	2.0	C_2/C_0	0	0.00763	0.005446	0.002904	0.000493
			0.0001	0.00762	0.005439	0.0029	0.000493
			0.0002	0.007611	0.005432	0.002897	0.000493
			0.0005	0.007582	0.005411	0.002886	0.000493
	3.0	C_2/C_0	0	0.123733	0.115546	0.103863	0.086128
			0.0001	0.123621	0.115441	0.103769	0.086051
			0.0002	0.123508	0.115336	0.103676	0.085974
			0.0005	0.123172	0.114987	0.103396	0.085745
	4.0	C_2/C_0	0	0.259256	0.249261	0.234724	0.211762
			0.0001	0.259069	0.249082	0.234557	0.211612
			0.0002	0.258883	0.248904	0.23439	0.211463
			0.0005	0.258326	0.24837	0.233891	0.211015

4. CONCLUSIONS

In this paper, the cost model, and the determination of optimal burn-in time to minimize a total mean cost for product sold under cumulative PRW warranty are studied. When the products have failure rate with the infant mortality period, that is, the probability that the product fails in the early period is greater than that in the useful period burn-in procedure must be taken into consideration to reduce manufacturing cost.

To determine optimal burn-in time, however, we need to consider burn-in and warranty costs under cumulative PRW. Thus, the total mean cost that is a sum of warranty and burn-in costs is calculated, and optimal burn-in time is obtained numerically. Then sensitivity analysis on cost parameters is studied. The main conclusion from numerical studies with specific distributions and restricted ranges are as follows:

For large lot size, short burn-in time is better. For very short or very long warranty period, the short burn-in time is better. Large values of cost parameters c_s enlarge optimal burn-in period but large values of cost parameters c_2, c_3 reduce optimal burn-in period.

REFERENCES

Blischke, W. R. and D. N. P. Murthy (1994). *Warranty Cost Analysis*, Marcel Dekker, New York.

Block, H. W. and T. H. Savits (1997). Burn-In, *Statistical Science*, **12**, 1-19.

Chou, K. and K. Tang (1992). Burn-in Time and Estimation of Change-Point with Weibull-Exponential Mixture Distribution, *Decision Science*, **23**, 973-990.

Kar, T. P. and J. A. Nachlas (1997). Coordinated Warranty and Burn-In Strategies, *IEEE Transactions on Reliability*, **46**, 512-518.

Kececioglu, D. B. and F. Sun (1997). *Burn-In: Its Quantification and Optimization*, Prentice Hall, New Jersey.

Kuo, W. and Y. Kuo (1983). Facing the Headaches of Early Failures: A State-of-the-Art Reviews of Burn-In Decisions, *Proceeding of The IEEE*, **71**, 1257-1266.

Leemis, L. M. and M. Beneke (1990). Burn-In Models and Methods: A Review, *IIE Transactions*, **22**, 172-180.

Mi, J. (1997). Warranty Policies and Burn-In, *Naval Research Logistics*, **44**, 199-209.

Mi, J. (1999). Comparisons of Renewable Warranties, *Naval Research Logistics*, **46**, 91-106.

Monga, A. and M. J. Zio (1998). Optimal System Design Considering Maintenance and Warranty, *Computers & Operations Research*, **25**, 691-705.

Nguyen, D. G. and D. N. P. Murthy (1982). Optimal Burn-In Time to Minimize Cost for Products Sold Under Warranty, *IIE Transactions*, **14**, 167-174.

Xie, M. (1989). On the Solution of Renewal-Type Integral Equations, *Communication in Statistics Simulation*, **18**, 281-293.