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Lifetime Estimation for Mixed Replacement Grouped Data in Competing Failures Model

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Abstract. The estimation of mean lifetimes in presence of interval censoring with mixed replacement procedure is examined when the distributions of lifetimes are exponential. It is assumed that, due to physical restrictions and/or economic constraints, the number of failures is investigated only at several inspection times during the lifetime test; thus there is interval censoring. The maximum likelihood estimator is found in an implicit form. The Cramer-Rao lower bound, which is the asymptotic variance of the estimator, is derived. The estimation of mean lifetimes for competing failures model has been expanded.

Key Words : *mixed replacement procedure, interval censoring, Cramer-Rao lower bound, exponential distribution, compound exponential distri bution.*

1. INTRODUCTION

The lifetime estimation of products requires a lot of time and cost with their developed reliability. In case of higher reliability products, especially, it may be impossible to observe the lifetime in the usual manner. Thus various methods such as censorings (Boardman, 1973) and accelerated testings (Nelson, 1990) are introduced in lifetime estimations. Lifetime estimating procedures with interval censoring procedures can be divided into two classes: With replacement - in which the failures are replaced at each inspection time and Without replacement (Wei and Bau, 1987) - in which the failures are not replaced. In general, it is well known that with replacement procedure is more accurate than without replacement one. In the with replacement procedure, however, one needs to prepare enough test items for replacement of failures at each inspection time. It is difficult to adopt the with replacement procedure in practice because one does not know exactly the number of items that will be required during the total time of test, which could

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be excessively large.

The practical difficulties in the with replacement procedure have the following two sides. First, the test may become too costly due to the overestimated number of failures. On the other hand, test items may run out before predetermined test time when the failures are underestimated. In the latter case, the test itself is terminated before predetermined test time.

In this paper an interval censoring model with mixed replacement procedure is examined which includes without replacement and with replacement procedure as its special cases. That is, we adopt with replacement procedure at the beginning of the test, and adopt without replacement one starting from the arbitrary but nonrandom inspection time. First, we try to find the estimator for mean lifetime with the mixed replacement procedure. And estimation for the case of competing failures is studied, subsequently.

2. PARAMETER ESTIMATION

2.1 Mixed Replacement Model

An interval censoring model with mixed replacement procedure in this paper may be described as follows. At first, n test items are placed on lifetime test and failures are observed at several arbitrary inspection times $\tau_j, j=1,\cdots,I$. At each inspection time, failures are replaced by new ones till the arbitrary inspection time τ_K which runs out test items for further replacement. The lifetime test is continued for the prespecified time T, so that

$$0=\tau_0<\tau_1<\cdots<\tau_K<\cdots<\tau_I=T .$$

Thus the mixed replacement procedure which is introduced in this paper can be the without replacement procedure if $K=0$ and can be the with replacement one if $K=I-1$.

Consider I test intervals Δ_j 's which have magnitude

$$\Delta_j=\tau_j-\tau_{j-1},\ j=1,\cdots,I \tag{2.1}$$

respectively.

Let the observed number of failures during the *jth* interval (2.1) be r_j and the number of test items unfailed till the last inspection time τ_I be r_{I+1}, then we have

$$r_{K+1}+r_{K+2}+\cdots+r_I+r_{I+1}=n .$$

Test items n_j which are placed at the beginning of the *jth* interval are all n in replaced intervals and are reduced with failures after ($K+1$)*th* inspection time, that is,

$$n_j=\begin{cases} n, & 1\le j\le K+1 \\ n-\sum_{i=K+1}^{j-1} r_i, & K+2\le j\le I \end{cases} .$$

2.2 Maximum Likelihood Estimator

The lifetimes of test items are assumed to be exponentially and independently distributed with density function

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0 \tag{2.2}$$

where θ is an unknown positive parameter.

The likelihood function for θ in interval censoring with mixed replacement procedure can be divided into two parts

$$L = L_1 \times L_2 \tag{2.3}$$

where L_1 is likelihood for replaced intervals and L_2 is likelihood for intervals not replaced.

The probabilities that failures happen in replaced intervals, that is, the probability that r_j failures are observed at the *jth* interval has the form of binomial distribution with parameters n and $1-e^{-\Delta_j/\theta}$ given as follows;

$$\frac{n!}{r_j!(n-r_j)!}(1-e^{-\frac{\Delta_j}{\theta}})^{r_j}(e^{-\frac{\Delta_j}{\theta}})^{n-r_j}.$$

So the likelihood function for replaced intervals is described

$$L_1 = \prod_{j=1}^{K} \frac{n!}{r_j!(n-r_j)!}(1-e^{-\frac{\Delta_j}{\theta}})^{r_j}(e^{-\frac{\Delta_j}{\theta}})^{n-r_j}. \tag{2.4}$$

The likelihood function for intervals not replaced has the form of multinomial distribution given as follows;

$$L_2 = \frac{n!}{r_{K+1}!\cdots r_{I+1}!}\prod_{j=K+1}^{I+1} P_j^{r_j} \tag{2.5}$$

where $P_j = e^{-\frac{\tau_{j-1}-\tau_K}{\theta}}(1-e^{-\frac{\Delta_j}{\theta}})$, $j = K+1, \cdots, I$ and $P_{I+1} = e^{-\frac{\tau_I-\tau_K}{\theta}}$.

According to (2.4) and (2.5), the likelihood function for mixed replacement procedure (2.3) becomes

$$L_1 = \prod_{j=1}^{K} \frac{n!}{r_j!(n-r_j)!}(1-e^{-\frac{\Delta_j}{\theta}})^{r_j}(e^{-\frac{\Delta_j}{\theta}})^{n-r_j} \frac{n!}{r_{K+1}!\cdots r_{I+1}!}\prod_{j=K+1}^{I+1} P_j^{r_j}. \tag{2.6}$$

Differentiating the log likelihood function with respect to θ, we get the first derivative

$$\begin{aligned}\frac{\partial \ln L}{\partial \theta} &= -\frac{1}{\theta^2}\sum_{j=1}^{I}\frac{r_j \Delta_j e^{-\frac{\Delta_j}{\theta}}}{1-e^{-\frac{\Delta_j}{\theta}}} + \frac{1}{\theta^2}\sum_{j=1}^{K}(n-r_j)\Delta_j \\ &+ \frac{1}{\theta^2}\sum_{j=K+1}^{I} r_j(\tau_{j-1}-\tau_K) + \frac{r_{I+1}(\tau_I-\tau_K)}{\theta^2}\end{aligned}. \tag{2.7}$$

Then the likelihood equation for estimating θ can be derived from (2.7) as

$$\sum_{j=1}^{I}\frac{r_j\Delta_j}{1-e^{-\frac{\Delta_j}{\hat{\theta}}}} - n\tau_K - \sum_{j=K+1}^{I} r_j(\tau_j-\tau_K) - r_{I+1}(\tau_I-\tau_K) = 0\,. \qquad (2.8)$$

Though the above maximum likelihood estimator has implicit form, it can be solved by using iterative methods such as Newton-Raphson's algorithm or EM algorithm suggested by Dempster et al.(1977) and it can be shown that the solution is unique.

2.3 Asymptotic Variance of the Estimator

The Cramer-Rao lower bound can be substituted for asymptotic variance, considering the form of the estimator derived earlier.

From the first derivative (2.7), we get the second derivative

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial\theta^2} = \frac{2}{\theta^3}\sum_{j=1}^{I}\frac{r_j\Delta_j}{1-e^{-\frac{\Delta_j}{\theta}}} - \frac{1}{\theta^4}\sum_{j=1}^{K}\frac{r_j\Delta_j^2 e^{-\frac{\Delta_j}{\theta}}}{(1-e^{-\frac{\Delta_j}{\theta}})^2} - \frac{2n\tau_K}{\theta^3} \\ -\frac{2}{\theta^3}\sum_{j=K+1}^{I} r_j(\tau_j-\tau_K) - \frac{2r_{I+1}(\tau_I-\tau_K)}{\theta^3}\end{aligned} \,. \qquad (2.9)$$

The expected value of r_j which is the observed number of failures during the jth interval and of r_{I+1} which is the number of unfailed items till τ_I, is given as follows;

$$E(r_j) = \begin{cases} n(1-e^{-\frac{\Delta_j}{\theta}}), & j=1,\cdots,K \\ ne^{-\frac{\tau_{j-1}-\tau_K}{\theta}}(1-e^{-\frac{\Delta_j}{\theta}}), & j=K+1,\cdots,I \end{cases}$$

and $$E(r_{I+1}) = ne^{-\frac{\tau_I-\tau_K}{\theta}}\,.$$

Then, the Cramer-Rao lower bound of the estimator becomes

$$CRLB(\hat{\theta}) = \frac{\theta^4}{n}\left[\sum_{j=1}^{K}\frac{\Delta_j^2}{e^{\frac{\Delta_j}{\theta}}-1} + \sum_{j=K+1}^{I}\frac{\Delta_j^2 e^{-\frac{\tau_{j-1}-\tau_K}{\theta}}}{e^{\frac{\Delta_j}{\theta}}-1}\right]^{-1}.$$

3. ESTIMATION IN COMPOUND EXPONENTIAL MODEL

We now derive the maximum likelihood estimator of competing failures model in interval censoring with mixed replacement procedure.

3.1 Competing Failures Model

The competing failures model means that there exist several failure modes. We assume that there are m failure modes in this paper. Then failures happen due to the first one of m failure modes. Through postmortem analysis, the observed number of failures during the *jth* interval can be decomposed as follows;

$$r_j = r_{1j} + \cdots + r_{mj} .$$

The lifetimes of test items are compound exponentially distributed with distribution function

$$F(x) = 1 - e^{-\frac{x}{\theta}}, \quad \frac{1}{\theta} = \sum_{i=1}^{m} \frac{1}{\theta_i}, \quad x > 0, \quad \theta_i > 0 \tag{3.1}$$

where θ_i is the unknown positive parameter of *ith* failure mode.

3.2 Maximum Likelihood Estimator

The probabilities that failure by *ith* failure mode happens at time x are derived

$$g_i(x) = f_i(x) \prod_{\substack{k=1 \\ k \neq i}}^{m} \{1 - F_k(x)\} = \frac{1}{\theta_i} e^{-\frac{x}{\theta}}$$

where $f_i(x) = \frac{1}{\theta_i} e^{-\frac{x}{\theta_i}}$.

So the probability that failure happens at time x regardless of failure modes is described

$$h(x) = \sum_{i=1}^{m} g_i(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} .$$

Now we consider the likelihood function which can be divided into two parts

$$L = L_1 \times L_2$$

where L_1 is likelihood for replaced intervals and L_2 is likelihood for intervals not replaced.

The probabilities that failures happen in replaced intervals, that is, the probability that r_{ij} failures by *ith* failure mode are observed at the *jth* interval has the form as follows;

$$P(r_{1j}, \cdots, r_{mj}; \theta_1, \cdots, \theta_m) = \frac{n!}{(n - r_j)!} \prod_{i=1}^{m} \frac{\{G_i(\Delta_j)\}^{r_{ij}}}{r_{ij}!} \{1 - H(\Delta_j)\}^{n - r_j} \tag{3.2}$$

where $G_i(\Delta_j) = \int g_i(x)dx = \frac{\theta}{\theta_i}(1 - e^{-\frac{\Delta_j}{\theta}})$ and $H(\Delta_j) = \sum_{i=1}^{m} G_i(\Delta_j) = 1 - e^{-\frac{\Delta_j}{\theta}}$.

So the likelihood function for replaced intervals becomes

$$L_1 = C_1 \prod_{j=1}^{K} (e^{-\frac{\Delta_j}{\theta}})^{n-r_j} \{\theta(1-e^{-\frac{\Delta_j}{\theta}})\}^{r_j} \prod_{i=1}^{m} (\frac{1}{\theta_i})^{r_{ij}}. \tag{3.3}$$

The likelihood function for intervals not replaced has the form of multinomial distribution given as follows;

$$L_2 = \prod_{j=K+1}^{I} \prod_{i=1}^{m} \frac{n!}{r_{ij}!(n-r)!} P_{ij}^{r_{ij}} \left\{1 - \sum_{i=1}^{m} G_i(\tau_I - \tau_K)\right\}^{n-r}$$

where the probability P_{ij} that failure by *ith* failure mode happens at *jth* interval is

$$P_{ij} = G_i(\tau_j - \tau_K) - G_i(\tau_{j-1} - \tau_K) = \frac{\theta}{\theta_i} e^{-\frac{\tau_j - \tau_K}{\theta}} (1 - e^{-\frac{\Delta_j}{\theta}}). \tag{3.4}$$

So the likelihood function for not replaced intervals is described

$$L_2 = C_2 \prod_{j=K+1}^{I} \prod_{i=1}^{m} \left\{\frac{\theta}{\theta_i} e^{-\frac{\tau_j - \tau_K}{\theta}} (1 - e^{-\frac{\Delta_j}{\theta}})\right\}^{r_{ij}} (e^{-\frac{\tau_I - \tau_K}{\theta}})^{n-r}. \tag{3.5}$$

According to (3.3) and (3.5), the likelihood function for competing failures model becomes

$$\begin{aligned} L = C \prod_{j=1}^{K} (e^{-\frac{\Delta_j}{\theta}})^{n-r_j} \{\theta(1-e^{-\frac{\Delta_j}{\theta}})\}^{r_j} \prod_{i=1}^{m} (\frac{1}{\theta_i})^{r_{ij}} \\ \prod_{j=K+1}^{I} \prod_{i=1}^{m} \left\{\frac{\theta}{\theta_i} e^{-\frac{\tau_j - \tau_K}{\theta}} (1 - e^{-\frac{\Delta_j}{\theta}})\right\}^{r_{ij}} (e^{-\frac{\tau_I - \tau_K}{\theta}})^{n-r} \end{aligned} \tag{3.6}$$

Differentiating the log likelihood function with respect to θ_i, we get the first derivative

$$\begin{aligned} \frac{\partial \ln L}{\partial \theta_i} = \frac{1}{\theta_i^2} \Bigg[\theta \sum_{j=1}^{I} r_j - \theta_i \sum_{j=1}^{I} r_{ij} + n\tau_K + \sum_{j=K+1}^{I} r_j(\tau_j - \tau_K) \\ + r_{I+1}(\tau_I - \tau_K) - \sum_{j=1}^{I} \frac{r_j \Delta_j}{1 - e^{-\frac{\Delta_j}{\theta}}} \Bigg]. \end{aligned} \tag{3.7}$$

Then the maximum likelihood estimator of θ_i is derived from (3.7) as

$$\hat{\theta}_i = \frac{r}{r_i} \hat{\theta}. \tag{3.8}$$

3.3 Cramer-Rao Lower Bound of the Estimator

Similarly, we can substitute the Cramer-Rao lower bound for asymptotic variance of the estimator.

We get the second derivative from the first derivative (3.7),

$$\frac{\partial^2 \ln L}{\partial \theta_i^2} = \frac{1}{\theta_i^4}\left[r\theta^2 - r_i\theta_i^2 - \sum_{j=1}^{I} \frac{r_j \Delta_j^2 e^{-\frac{\Delta_j}{\theta}}}{(1-e^{-\frac{\Delta_j}{\theta}})^2} \right]$$

$$- \frac{2}{\theta_i^3}\left[r\theta - r_i\theta_i + n\tau_K + \sum_{j=K+1}^{I} r_j(\tau_j - \tau_K) + r_{I+1}(\tau_I - \tau_K) - \sum_{j=1}^{I} \frac{r_j\Delta_j}{1-e^{-\frac{\Delta_j}{\theta}}} \right]. \quad (3.9)$$

The expected value of r_{ij} which is the observed number of failures by the *ith* failure mode during the *jth* interval is given as follows;

$$E(r_{ij}) = \begin{cases} n\dfrac{\theta}{\theta_i}(1-e^{-\frac{\Delta_j}{\theta}}), & j = 1,\cdots,K \\ n\dfrac{\theta}{\theta_i}e^{-\frac{\tau_{j-1}-\tau_K}{\theta}}(1-e^{-\frac{\Delta_j}{\theta}}), & j = K+1,\cdots,I \end{cases}.$$

And the expected value of r_{I+1} which is the number of unfailed items till τ_I is given

$$E(r_{I+1}) = ne^{-\frac{\tau_I - \tau_K}{\theta}}.$$

Also, the expected value of r_j which is the observed number of failures during the *jth* interval regardless of failure modes and of r_i which is the observed number of failures by the *ith* failure mode through the total test time, is descried as follows;

$$E(r_j) = \begin{cases} n(1-e^{-\frac{\Delta_j}{\theta}}), & j = 1,\cdots,K \\ ne^{-\frac{\tau_{j-1}-\tau_K}{\theta}}(1-e^{-\frac{\Delta_j}{\theta}}), & j = K+1,\cdots,I \end{cases}$$

and

$$E(r_i) = n\frac{\theta}{\theta_i}\left\{ \sum_{j=1}^{K}(1-e^{-\frac{\Delta_j}{\theta}}) + \sum_{j=K+1}^{I} e^{-\frac{\tau_{j-1}-\tau_K}{\theta}}(1-e^{-\frac{\Delta_j}{\theta}}) \right\}.$$

Hence, we get the expected value of r which is the observed number of total failures

$$E(r) = n\left\{ \sum_{j=1}^{K}(1-e^{-\frac{\Delta_j}{\theta}}) + \sum_{j=K+1}^{I} e^{-\frac{\tau_{j-1}-\tau_K}{\theta}}(1-e^{-\frac{\Delta_j}{\theta}}) \right\}.$$

The Cramer-Rao lower bound of the estimator is derived

$$CRLB(\hat{\theta}_i) = \frac{\theta_i^4}{n}\left[\theta(\theta_i - \theta)\left\{ \sum_{j=1}^{K}(1-e^{-\frac{\Delta_j}{\theta}}) + \sum_{j=K+1}^{I} e^{-\frac{\tau_{j-1}-\tau_K}{\theta}}(1-e^{-\frac{\Delta_j}{\theta}}) \right\} \right.$$

$$+\sum_{j=1}^{K}\frac{\Delta_j^2}{e^{\frac{\Delta_j}{\theta}}-1}+\sum_{j=K+1}^{I}\frac{\Delta_j^2 e^{-\frac{\tau_{j-1}-\tau_K}{\theta}}}{e^{\frac{\Delta_j}{\theta}}-1}\Bigg]^{-1}.$$

4. SUMMARY

The mixed replacement procedure studied in this paper can be useful in practical lifetime estimation because of its convenient and flexible scheme. As a special case, it can be a without replacement procedure or a with replacement one. The maximum likelihood estimator is obtainable. The Cramer-Rao lower bound is derived instead of variance, considering the implicit form of estimator. Finally, the estimation of mean lifetimes for competing failures model has been expanded.

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