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Bayesian Estimation for the Reliability of Stress-Strength Systems Using Noninformative Priors

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Abstract. Consider the problem of estimating the system reliability using noninformative priors when both stress and strength follow generalized gamma distributions. We first treat the orthogonal reparametrization and then, using this reparametrization, derive Jeffreys' prior, reference prior, and matching priors. We next provide the sufficient condition for propriety of posterior distributions under those noninformative priors. Finally, we provide and compare estimated values of the system reliability based on the simulated values of the parameter of interest in some special cases.

Key Words : *System reliability, generalized gamma distribution, orthogonal reparametrization, Jeffreys' prior, reference prior, matching prior, posterior.*

1. INTRODUCTION

Suppose a system, made up of k identical components, functions if r or more of the k components simultaneously operate. We assume that the strengths of these components, $Y_1, \cdots, Y_k$, are identically and independently distributed (i.i.d.) random variables with a common commulative distribution function (c.d.f.), $G(y)$. We further suppose that this system is subject to a stress, say X, which is a random variable with c.d.f., $F(x)$. The system operates satisfactorily if r or more of the k components have strength large then the stress, X, and accordingly, we define the system reliability , $R_{r,k}$ say , as the probability that at least r of $Y_1, \cdots, Y_k$ exceed X , so that

$$R_{r,k} = \sum_{i=r}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [1 - G(x)]^i [G(x)]^{k-i} dF(x). \tag{1.1}$$

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The problem of making inference about (1.1) has been discussed, using the classical frequentist theory approach, in various guises by Birnbaum(1956), Bhattacharyya and Johnson(1974), Simonoff et al.(1986), and Reiser and Guttman(1989), among others. A great deal of this work have focused on producing maximum likelihood estimatous, uniformly minimum variance unbiased estimators, and one-sided confidence intervals for $R_{r,k}$ in various situations. In contrast, there is relatively little on a Bayesian approach to this problem. Some pertient references are Draper and Guttman(1978), Gaithwaite and Dickey(1988), Guttman et al.(1990) , and Guttman and Papandonatos(1997).

The present paper focuses exclusively on Bayesian inference for $R_{r,k}$ when F(x) and G(y) are c.d.f.'s of generalized gamma distributions $GG(\eta_1, \beta, p)$ and $GG(\eta_2, \beta, p)$ respectively, with corresponding density functions

$$f(x) = \frac{\beta}{\Gamma(p)} {\eta_1}^{-p\beta} x^{p\beta-1} e^{-(\frac{x}{\eta_1})^\beta}, x > 0$$

and

$$g(y) = \frac{\beta}{\Gamma(p)} {\eta_2}^{-p\beta} y^{p\beta-1} e^{-(\frac{y}{\eta_2})^\beta}, y > 0$$

with $\eta_1 > 0$, $\eta_2 > 0$, $\beta > 0$, and $p > 0$. In this situation, the system reliability $R_{r,k}$ in (1.1) reduces, after some manipulation, to

$$R_{r,k} = \sum_{i=r}^{k} \binom{k}{i} \int_0^\infty [1 - I(p,u)]^i [I(p,u)]^{k-i} \frac{1}{\Gamma(p)} {\theta_1}^p u^{p-1} e^{-\theta_1 u} du, \qquad (1.2)$$

where $\theta_1 = (\frac{\eta_2}{\eta_1})^\beta$ and $I(p,u) = \int_0^u \frac{1}{\Gamma(p)} v^{p-1} e^{-v} dv$.

In the generalized gamma distribution $GG(\eta, \beta, p)$, η,β, and p are , respectively, called the scale parameter, the shape parameter, and the index parameter. This distribution includes many interesting distributions as special cases : exponential distribution ($p = \beta = 1$), Raleigh distribution ($p = 1, \beta = 2$),Weibull distribution (p=1), Maxwell distribution ($p = \frac{3}{2}, \beta = 2$), half-normal distribution($p = \frac{1}{2}, \beta = 2$) and gamma distribution ($\beta = 1$).

In this paper we only consider the case when p is known. Since $R_{r,k}$ in (1.2) depends only on θ_1, the emphasis is on noninformative priors for θ_1.

The most frequently used noninformative prior is Jeffreys'(1961) prior, which is proportional to the positive square root of the determinant of the Fisher information matrix. In spite of its success in one-parameter problems, Jeffreys' prior frequently runs into serious difficulties in the presence of nuisance parameter. As an alternative, we use the method of Peers(1965) to find priors which require the frequentist coverage probability of the posterior region of a real-values parametric function to match the nominal level with a remainder of $O(n^{-1})$. Tibshirani(1989) reconsidered the case when the real-valued parameter is orthogonal to the nuisance parameter vector in the sense of Cox and Reid(1987). These priors, as usually referred to as matching priors, were further studied in Datta and Ghosh(1995).

On the other hand, Berger and Bernardo(1989,1992), and Datta and Ghosh(1995) extended Bernardo's(1979) reference-prior approach, giving a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. This approach is very successful in various practical problems. For a list of recent reviews of various approaches to the development of noninformative priors, we refer to Kass and Wasserman(1996).

In the case of $k = 1$, Thompson and Basu(1993) derived reference prior for $R_{1,1}$ when the stress and strength are both exponentially distributed. It turns out that in such situations. The reference priors agree with Jeffreys' prior. Recently, when $k = 1$, Sun et al.(1998) derived matching priors as well as reference priors for $R_{1,1}$ when both stress and strength follow Weibull distribution.

In this paper we derive matching priors as well as reference priors for θ_1 and consider the problem of estimating $R_{r,k}$ in generalized gamma stress-strength models when p is known. Section 2 treats the orthogonal reparameterization from (η_1, η_2, β) to $(\theta_1, \theta_2, \theta_3)$ when θ_1 is the parameter of interest and (θ_2, θ_3) is a nuisance parameter vector. In Section 3, we derive, using this orthogonal reparameterization, Jeffreys' prior, reference prior , and matching priors when θ_1 is the parameter of interest. The sufficient condition for propriety of posterior distributions of $(\theta_1, \theta_2, \theta_3)$ and marginal posterior densities of θ_1 under these priors are given in Section 4. In Section 5, we provide and compare estimated values of $R_{r,k}$ based on the simulated values of θ_1 by Gibbs sampler for several pairs of k and r when $p = 1$.

2. FISHER INFORMATION MATRICES

2.1 Original Parametrization

Suppose that $X_1, \ldots, X_m$ are i.i.d. as the generalized gamma distribution, $GG(\eta_1, \beta, p)$ and independently, $Y_1, \cdots, Y_n$ are i.i.d. as $GG(\eta_2, \beta, p)$.

Then the likelihood function of (η_1, η_2, β) is

$$L(\eta_1, \eta_2, \beta|\underline{x}, \underline{y}) = \beta^{m+n}[\Gamma(p)]^{-(m+n)}\eta_1^{-mp\beta}\eta_2^{-np\beta}\Big(\prod_{i=1}^{m} x_i \prod_{j=1}^{n} y_j\Big)^{p\beta-1} \cdot e^{-\sum_{i=1}^{m}(\frac{x_i}{\eta_1})^\beta - \sum_{j=1}^{n}(\frac{y_j}{\eta_2})^\beta} \quad (2.1)$$

and the log-likelihood function of (η_1, η_2, β) is

$$\begin{aligned} l(\eta_1, \eta_2, \beta|\underline{x}, \underline{y}) &= logL(\eta_1, \eta_2, \beta|\underline{x}, \underline{y}) \\ &\propto (m+n)log\beta - mp\beta log\eta_1 - np\beta log\eta_2 \\ &+ (p\beta - 1)(\sum_{i=1}^{m} \log x_i + \sum_{j=1}^{n} \log y_j) - \sum_{i=1}^{m}(\frac{x_i}{\eta_1})^\beta - \sum_{j=1}^{n}(\frac{y_j}{\eta_2})^\beta. \end{aligned}$$

Lemma 2.1. The Fisher information matrix for (η_1, η_2, β) is

$$I_1(\eta_1, \eta_2, \beta) = \begin{pmatrix} \frac{mp\beta^2}{\eta_1^2} & 0 & -\frac{m\gamma_1}{\eta_1\Gamma(p)} \\ 0 & \frac{mp\beta^2}{\eta_2^2} & -\frac{n\gamma_1}{\eta_2\Gamma(p)} \\ -\frac{m\gamma_1}{\eta_1\Gamma(p)} & -\frac{n\gamma_1}{\eta_2\Gamma(p)} & \frac{(m+n)}{\beta^2}(1+\frac{\gamma_2}{\Gamma(p)}) \end{pmatrix}, \tag{2.2}$$

where $\gamma_i = \int_0^\infty (logz)^i z^p e^{-z} dz, i = 1, 2.$

Proof. The result easily follows from the following identities :

$$\begin{aligned}
\frac{\partial^2 l}{\partial \eta_1^2} &= \frac{mp\beta}{\eta_1^2} - \beta(\beta+1)\sum_{i=1}^{m} \frac{x_i{}^{\beta}}{\eta_1{}^{\beta+2}}, \\
\frac{\partial^2 l}{\partial \eta_1 \partial \eta_2} &= 0, \\
\frac{\partial^2 l}{\partial \eta_1 \partial \beta} &= -\frac{mp}{\eta_1} + \frac{1}{\eta_1}\sum_{i=1}^{m}\Big[\Big(\frac{x_i}{\eta_1}\Big)^{\beta} + \beta\Big(\frac{x_i}{\eta_1}\Big)^{\beta} log\Big(\frac{x_i}{\eta_1}\Big)\Big], \\
\frac{\partial^2 l}{\partial \eta_2^2} &= \frac{np\beta}{\eta_2^2} - \beta(\beta+1)\sum_{j=1}^{n} \frac{y_j{}^{\beta}}{\eta_2{}^{\beta+2}}, \\
\frac{\partial^2 l}{\partial \eta_2 \partial \beta} &= -\frac{np}{\eta_2} + \frac{1}{\eta_2}\sum_{j=1}^{n}\Big[\Big(\frac{y_j}{\eta_2}\Big)^{\beta} + \beta\Big(\frac{y_j}{\eta_2}\Big)^{\beta} log\Big(\frac{y_j}{\eta_2}\Big)\Big], \\
\frac{\partial^2 l}{\partial \beta^2} &= -\frac{m+n}{\beta^2} - \sum_{i=1}^{m}\Big(\frac{x_i}{\eta_1}\Big)^{\beta}(log\Big(\frac{x_i}{\eta_1}\Big))^2 - \sum_{j=1}^{n}\Big(\frac{y_j}{\eta_2}\Big)^{\beta}(log\Big(\frac{y_j}{\eta_2}\Big))^2,
\end{aligned}$$

and

$$\begin{aligned}
E(X_i{}^{\beta}) &= p\eta_1{}^{\beta}, \qquad E(Y_j{}^{\beta}) = p\eta_2{}^{\beta}, \\
E(X_i{}^{\beta} logX_i) &= p\eta_1{}^{\beta}(log\eta_1) + \frac{1}{\beta\Gamma(p)}\eta_1{}^{\beta}\gamma_1, \\
E(Y_j{}^{\beta} logY_j) &= p\eta_2{}^{\beta}(log\eta_2) + \frac{1}{\beta\Gamma(p)}\eta_2{}^{\beta}\gamma_1, \\
E(X_i{}^{\beta}(logX_i)^2) &= p\eta_1{}^{\beta}(log\eta_1)^2 + \frac{2}{\beta\Gamma(p)}\eta_1{}^{\beta}(log\eta_1)\gamma_1 + \frac{1}{\beta^2\Gamma(p)}\eta_1{}^{\beta}\gamma_2, \\
E(Y_j{}^{\beta}(logY_j)^2) &= p\eta_2{}^{\beta}(log\eta_2)^2 + \frac{2}{\beta\Gamma(p)}\eta_2{}^{\beta}(log\eta_2)\gamma_1 + \frac{1}{\beta^2\Gamma(p)}\eta_2{}^{\beta}\gamma_2.
\end{aligned}$$

2.2 Orthogonal Reparametrization

Consider the following transformation from (η_1, η_2, β) to $(\theta_1, \phi_2, \phi_2)$:

$$\Big(\frac{\eta_2}{\eta_1}\Big)^{\beta} = \theta_1, \eta_1 = \phi_1 \equiv \phi_1(\theta_1, \theta_2, \theta_3), \beta = \phi_2 \equiv \phi_1(\theta_1, \theta_2, \theta_3).$$

Then the Fisher information matrix for $(\theta_1, \phi_1, \phi_2)$ becomes, after some calculations,

$$I_2(\theta_1, \phi_1, \phi_2) = \begin{pmatrix} i_{\theta_1\theta_1} & i_{\theta_1\phi_1} & i_{\theta_1\phi_2} \\ i_{\phi_1\theta_1} & i_{\phi_1\phi_1} & i_{\phi_1\phi_2} \\ i_{\phi_2\theta_1} & i_{\phi_2\phi_1} & i_{\phi_2\phi_2} \end{pmatrix}, \qquad (2.3)$$

where

$$\begin{aligned} i_{\theta_1\theta_1} &= np\theta_1^{-2}, \\ i_{\theta_1\phi_1} &= \frac{np\phi_2}{\phi_1\theta_1} = i_{\phi_1\theta_1}, \\ i_{\theta_1\phi_2} &= -\frac{np(log\theta_1)}{\phi_2\theta_1} - \frac{n\gamma_1}{\phi_2\Gamma(p)\theta_1} = i_{\phi_2\theta_1}, \\ i_{\phi_1\phi_1} &= \frac{(m+n)p\phi_2^2}{\phi_1^2}, \\ i_{\phi_1\phi_2} &= -\frac{np(log\theta_1)}{\phi_1} - \frac{(m+n)\gamma_1}{\phi_1\Gamma(p)} = i_{\phi_2\phi_1}, \end{aligned}$$

and

$$i_{\phi_2\phi_2} = \frac{np(log\theta_1^2)}{\phi_2^2} + \frac{2n\gamma_1 log\theta_1}{\phi_2^2\Gamma(p)} + \frac{m+n}{\phi_2^2}(1 + \frac{\gamma_2}{\Gamma(p)}).$$

Following Cox Reid's(1987) method to the parameter orthogonalization, we have

$$\sum_{r=1}^{2} i_{\phi_r\phi_s}\frac{\partial\phi_r}{\partial\theta_1} = -i_{\theta_1\phi_s}, \qquad s = 1, 2,$$

that is, from (2.3) ,

$$\frac{(m+n)p\phi_2^2}{\phi_1^2}\frac{\partial\phi_1}{\partial\theta_1} - \Big[\frac{(m+n)\gamma_1}{\phi_1\Gamma(p)} + \frac{nplog\theta_1}{\phi_1}\Big]\frac{\partial\phi_2}{\partial\theta_1} = -\frac{np\phi_2}{\phi_1\theta_1}$$

and

$$\begin{aligned} -\Big[\frac{np(log\theta_1)}{\phi_1} + \frac{(m+n)\gamma_1}{\phi_1\Gamma(p)}\Big]\frac{\partial\phi_1}{\partial\theta_1} + \Big[\frac{np(log\theta_1)^2}{\phi_2^2} + \frac{2n\gamma_1 log\theta_1}{\phi_2^2\Gamma(p)} \\ + \frac{m+n}{\phi_2^2}(1 + \frac{\gamma_2}{\Gamma(p)})\Big]\frac{\partial\phi_2}{\partial\theta_1} = \frac{np(log\theta_1)}{\phi_2\theta_1} + \frac{n\gamma_1}{\phi_2\Gamma(p)\theta_1}. \end{aligned}$$

After some manipulation, we get

$$\frac{1}{\phi_1}\frac{\partial\phi_1}{\partial\theta_1} = \frac{1}{a_2(\theta_2, \theta_3)[mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^{\frac{3}{2}}} \cdot \frac{1}{\theta_1}\Big[\frac{mn\gamma_1 log\theta_1}{\Gamma(p)} - (m+n)n\gamma_*\Big]$$

and

$$\frac{1}{\phi_2}\frac{\partial\phi_2}{\partial\theta_1} = \frac{mnp(log\theta_1)\theta_1^{-1}}{mnp(log\theta_1)^2+(m+n)^2\gamma_*}, \tag{2.4}$$

where $\gamma_* = 1 + \frac{\gamma_2}{\Gamma(p)} - \frac{\gamma_1^2}{p\Gamma^2(p)}$.
Then the general solutions of (2.4) with respect to (ϕ_1, ϕ_2) become

$$\phi_1 = a_1(\theta_2,\theta_3)\theta_1^{-\frac{n}{m+n}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}} \cdot e^{-\frac{\gamma_1}{p\Gamma(p)}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}}$$

and

$$\phi_2 = a_2(\theta_2,\theta_3)\sqrt{mnp(log\theta_1)^2+(m+n)^2\gamma_*}. \tag{2.5}$$

Now, choosung $a_1(\theta_2,\theta_3) = \theta_2$ and $a_2(\theta_2,\theta_3) = \theta_3$ in (2.5) and using the inverse transformation

$$\eta_1 = \phi_1, \eta_2 = \theta_1^{\frac{1}{\phi_2}}\phi_1, \beta = \phi_2,$$

we have, from (2.5) ,

$$\eta_1 = \theta_2\theta_1^{-\frac{n}{m+n}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}} \cdot e^{-\frac{r_1}{p\Gamma(p)}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}},$$
$$\eta_2 = \theta_2\theta_1^{\frac{m}{m+n}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}} \cdot e^{-\frac{r_1}{p\Gamma(p)}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}},$$

and

$$\beta = \theta_3[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{\frac{1}{2}}. \tag{2.6}$$

Lemma 2.2. The Fisher information matrix for $(\theta_1, \theta_2, \theta_3)$ is

$$I_3(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} i_{11} & 0 & 0 \\ 0 & i_{22} & 0 \\ 0 & 0 & i_{33} \end{pmatrix}, \tag{2.7}$$

where

$$i_{11} = mn(m+n)p\gamma_*\theta_1^{-2}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-1},$$
$$i_{22} = (m+n)p\theta_2^{-2}\theta_3^2[mnp(log\theta_1)^2+(m+n)^2\gamma_*],$$

and

$$i_{33} = \frac{1}{m+n}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]\theta_3^{-2}.$$

Proof. The result follows, after lengthy calculations using (2.6), from

$$I_3(\theta_1,\theta_2,\theta_3) = J^T I_1(\theta_1,\theta_2,\theta_3)J,$$

where $I_1(\theta_1,\theta_2,\theta_3)$ is $I_1(\eta_1,\eta_2,\beta)$ in (2.2) expressed in terms of $(\theta_1,\theta_2,\theta_3)$ and

$$J = (J_{ij}) \ , \ i = 1,2,3 \ , j = 1,2,3$$

with

$$\begin{aligned}
J_{11} &= \frac{\partial \eta_1}{\partial \theta_1} \\
&= \theta_1^{-\frac{n}{m+n}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}-1} \cdot \theta_2 \cdot \theta_3^{-1} \\
&\quad \cdot [mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^{-\frac{3}{2}} \cdot \left\{ \frac{mn\gamma_1}{\Gamma(p)} log\theta_1 - (m+n)n\gamma_* \right\} \\
&\quad \cdot e^{-\frac{\gamma_1}{p\Gamma(p)}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}}, \\
J_{12} &= \frac{\partial \eta_1}{\partial \theta_2} = \theta_1^{-\frac{n}{m+n}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}} \cdot e^{-\frac{\gamma_1}{p\Gamma(p)}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}}, \\
J_{13} &= \frac{\partial \eta_1}{\partial \theta_3} = \theta_1^{-\frac{n}{m+n}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}} \cdot \theta_2 \cdot \theta_3^{-2} \\
&\quad \cdot [mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^{-\frac{1}{2}} \cdot \left\{ \frac{n}{m+n} log\theta_1 + \frac{\gamma_1}{p\Gamma(p)} \right\} \\
&\quad \cdot e^{-\frac{\gamma_1}{p\Gamma(p)}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}}, \\
J_{21} &= \frac{\partial \eta_2}{\partial \theta_1} \\
&= \theta_1^{\frac{m}{m+n}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}-1} \cdot \theta_2 \cdot \theta_3^{-1} \\
&\quad \cdot [mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^{-\frac{3}{2}} \cdot \left\{ m(m+n)\gamma_* + \frac{mn\gamma_1}{\Gamma(p)} log\theta_1 \right\} \\
&\quad \cdot e^{-\frac{\gamma_1}{p\Gamma(p)}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}}, \\
J_{22} &= \frac{\partial \eta_2}{\partial \theta_2} = \theta_1^{\frac{m}{m+n}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}} \cdot e^{-\frac{\gamma_1}{p\Gamma(p)}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}} \\
J_{23} &= \frac{\partial \eta_2}{\partial \theta_3} = \theta_1^{\frac{m}{m+n}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}} \cdot \theta_2 \cdot \theta_3^{-2} \\
&\quad \cdot [mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^{-\frac{1}{2}} \cdot \left\{ \frac{\gamma_1}{p\Gamma(p)} - \frac{m}{m+n} log\theta_1 \right\} \\
&\quad \cdot e^{-\frac{\gamma_1}{p\Gamma(p)}\frac{1}{\theta_3}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}}, \\
J_{31} &= \frac{\partial \beta}{\partial \theta_1} = mnp(log\theta_1)\theta_1^{-1}\theta_3 \cdot [mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^{-\frac{1}{2}}, \\
J_{32} &= \frac{\partial \beta}{\partial \theta_2} = 0,
\end{aligned}$$

and

$$J_{33} = \frac{\partial \beta}{\partial \theta_3} = [mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^{\frac{1}{2}}.$$

This implies that θ_1 , the parameter of interest, is orthogonal to the nuisance parameter vector (θ_2, θ_3) in the sence of Cox and Reid(1987).

3. NONINFORMATIVE PRIORS

In this section, we provide, using (2.7), three types of noninformative priors ; Jeffreys' prior, reference priors, and matching priors.

Theorem 3.1. The Jeffreys' prior for $(\theta_1, \theta_2, \theta_3)$ is

$$\begin{aligned} \pi_J(\theta_1, \theta_2, \theta_3) &\propto \mid I_3(\theta_1, \theta_2, \theta_3) \mid^{\frac{1}{2}} \\ &\propto \theta_1^{-1}\theta_2^{-1}[mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^{\frac{1}{2}}. \end{aligned} \tag{3.1}$$

From Datta and Ghosh(1995), we have the reference prior for $(\theta_1, \theta_2, \theta_3)$ in the following when θ_1 is the parameter of interest and (θ_2, θ_3) is the nuisance parameter vector.

Theorem 3.2. The reference prior for θ_1 is given by

$$\pi_R(\theta_1, \theta_2, \theta_3) \propto \theta_1^{-1}\theta_2^{-1}[mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^{-\frac{1}{2}}. \tag{3.2}$$

Proof. The Fisher information matrix $I_3(\theta_1, \theta_2, \theta_3)$ is

$$I_3(\theta_1, \theta_2, \theta_3) = \ block\ diagonal\ (h_1(\theta_1, \theta_2, \theta_3)\ ,\ H_2(\theta_1, \theta_2, \theta_3)),$$

where $h_1(\theta_1, \theta_2, \theta_3) = i_{11}$ and $H_2(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} i_{22} & 0 \\ 0 & i_{33} \end{pmatrix}$

Then $h_1(\theta_1, \theta_2, \theta_3) = h_{11}(\theta_1)h_{12}(\theta_2, \theta_3)$ and $|H_2(\theta_1, \theta_2, \theta_3)| = h_{21}(\theta_2, \theta_3)h_{22}(\theta_1)$ with $h_{11}(\theta_1) = mn(m+n)p\gamma_*\theta_1^{-2}\cdot[mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^{-1}$, $h_{12}(\theta_2, \theta_3) = 1$, $h_{21}(\theta_2, \theta_3) = p\theta_2^{-2}$, and $h_{22}(\theta_1) = [mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^2$. Now, take the sequence $\{\Omega_i\}$ of retangular compact subsets of $\Omega = \{(\theta_1, \theta_2, \theta_3) \mid 0 < \theta_1, \theta_2, \theta_3 < \infty\}$ where $\Omega_i = A_1^i \times A_2^i$ with $A_1^i = \left[\frac{1}{i}, i\right]$, an increasing compact sets for θ_1 , and $A_2^i = \left[\frac{1}{i}, i\right] \times \left[\frac{1}{i}, i\right]$, an increasing compact sets for (θ_2, θ_3). Then we have , from Datta and Ghosh(1995), the reference prior for $(\theta_1, \theta_2, \theta_3)$,

$$\begin{aligned} \pi_R(\theta_1, \theta_2, \theta_3) &= [h_{11}(\theta_1)h_{21}(\theta_2, \theta_3)]^{\frac{1}{2}} \\ &\propto \theta_1^{-1}\theta_2^{-1}[mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^{-\frac{1}{2}}. \end{aligned}$$

Also, following Tibshirani(1989), we have matching priors for θ_1, the parameter of interest. as follows :

Theorem 3.3. The matching priors for θ_1 are given by

$$\begin{aligned}\pi_M(\theta_1,\theta_2,\theta_3) &\propto i_{11}^{\frac{1}{2}} g(\theta_2,\theta_3) \\ &\propto \theta_1^{-1}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}} g(\theta_2,\theta_3) \qquad (3.3)\end{aligned}$$

for any positive differentiable function g.

Note that Jeffreys' prior is not a matching prior, but the reference prior is a matching prior with $g(s,t)=\frac{1}{s}$ in (3.3). An interesting class of matching priors can be obtained by taking $g(s,t)=s^{-1}t^{-b}$ for $-\infty<b<\infty$, for which (3.3) becomes

$$\pi_M(\theta_1,\theta_2,\theta_3)\propto \theta_1^{-1}\theta_2^{-1}\theta_3^{-b}[mnp(log\theta_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}. \qquad (3.4)$$

4. POSTERIOR DISTRIBUTIONS

The posterior density of $(\theta_1,\theta_2,\theta_3)$ under a prior π is

$$\pi(\theta_1,\theta_2,\theta_3 \mid \underline{x},\underline{y}) \propto L(\theta_1,\theta_2,\theta_3 \mid \underline{x},\underline{y})\pi(\theta_1,\theta_2,\theta_3), \qquad (4.1)$$

where $L(\theta_1,\theta_2,\theta_3 \mid \underline{x},\underline{y})$ is the likelihood function $L(\eta_1,\eta_2,\beta \mid \underline{x},\underline{y})$ in (2.1) expressed in terms of

$$\theta_1=(\frac{\eta_2}{\eta_1})^\beta \quad , \quad \theta_2=\eta_1^{\frac{m}{m+n}}\eta_2^{\frac{n}{m+n}}e^{\frac{\gamma_1}{p\Gamma(p)}\frac{1}{\beta}},$$
$$\theta_3=\beta[mnp(\beta log\frac{\eta_2}{\eta_1})^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}.$$

We first provide the sufficient condition under which the posteriors are proper under π_J in (3.1), π_R in (3.2), and π_M for $b\le 2$ in (3.4). Note that for almost all samples from a continuous distribution, observations are distinct.

Theorem 4.1. All the posteriors under π_J, π_R and π_M for $b\le 2$ in (3.4) are proper if $m+n\ge 3$. **Proof**. We only prove the result for π_M. Using the transformation

$$\begin{aligned}\theta_1 &= u_1, \\ \theta_2 &= u_1^{\frac{n}{m+n}\frac{1}{u_3}} u_2^{\frac{1}{u_3}} e^{\frac{\gamma_1}{p\Gamma(p)}\frac{1}{u_3}}, \\ \theta_3 &= u_3[mnp(logu_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}},\end{aligned}$$

and Jacobian $J=u_1^{\frac{n}{m+n}\frac{1}{u_3}} u_2^{\frac{1}{u_3}-1} e^{\frac{\gamma_1}{p\Gamma(p)}\frac{1}{u_3}} u_3^{-1}[mnp(logu_1)^2+(m+n)^2\gamma_*]^{-\frac{1}{2}}$,

$$\int_0^\infty\int_0^\infty\int_0^\infty L(\theta_1,\theta_2,\theta_3 \mid \underline{x},\underline{y})\pi_M(\theta_1,\theta_2,\theta_3)d\theta_1 d\theta_2 d\theta_3$$
$$=\int_0^\infty\int_0^\infty\int_0^\infty u_3^{m+n}[\Gamma(p)]^{-(m+n)}u_1^{-np}u_2^{-(m+n)p}(\prod_{i=1}^m x_i\prod_{j=1}^n y_j)^{pu_3-1}$$

$$\cdot e^{-\frac{1}{u_2}[\sum_{i=1}^m x_i^{u_3}+\frac{1}{u_1}\sum_{j=1}^n y_j^{u_3}]} u_1^{-1}u_2^{-1}u_3^{-b-1}[mnp(logu_1)^2+(m+n)^2\gamma_*]^{\frac{b}{2}-1}du_1du_2du_3$$

$$= \int_0^\infty\int_0^\infty [\Gamma(p)]^{-(m+n)}\Gamma[(m+n)p]u_1^{mp-1}u_3^{m+n-b-1}[u_1\sum_{i=1}^m x_i^{u_3}+\sum_{j=1}^n y_j^{u_3}]^{-(m+n)p}$$

$$\cdot(\prod_{i=1}^m x_i\prod_{j=1}^n y_j)^{pu_3-1}[mnp(logu_1)^2+(m+n)^2\gamma_*]^{\frac{b}{2}-1}du_3du_1$$

$$= \int_0^\infty [\Gamma(p)]^{-(m+n)}\cdot\Gamma((m+n)p)(\prod_{i=1}^m x_i\prod_{j=1}^n y_j)^{pu_3-1}u_3^{m+n-b-1}(\sum_{i=1}^m x_i^{u_3})^{-mp}(\sum_{j=1}^n y_j^{u_3})^{-np}$$

$$\cdot\left(\int_0^\infty v^{mp-1}(1+v)^{-(m+n)p}[mnp(log\frac{\sum_{j=1}^n y_j^{u_3}}{\sum_{i=1}^m x_i^{u_3}}v)^2+(m+n)^2\gamma_*]^{\frac{b}{2}-1}dv\right)du_3. \quad (4.2)$$

Now, it can be easily shown that for $b \le 2$,

$$h(v) = [mnp(log\frac{\sum_{j=1}^n y_j^{u_3}}{\sum_{i=1}^m x_i^{u_3}}+logv)^2+(m+n)^2\gamma_*]^{\frac{b}{2}-1}$$

has a maximum value $[(m+n)^2\gamma_*]^{\frac{b}{2}-1}$ at $v = \frac{\sum_{i=1}^m x_i^{u_3}}{\sum_{j=1}^n y_j^{u_3}}$. Hence

$$(4.2) \le [(m+n)^2\gamma_*]^{\frac{b}{2}-1}[\Gamma(p)]^{-(m+n)}\Gamma(mp)\Gamma(np)$$

$$\cdot\int_0^\infty(\prod_{i=1}^m x_i\prod_{j=1}^n y_j)^{pu_3-1}u_3^{m+n-b-1}(\sum_{i=1}^m x_i^{u_3})^{-mp}(\sum_{j=1}^n y_j^{u_3})^{-np}du_3. \quad (4.3)$$

Since there exist x_k and y_l such that $x_k < max\{x_1,x_2,\cdots,x_m\}$ and $y_l < max\{y_1,y_2,\cdots,y_n\}$, we have

$$rhs\ of\ (4.3) \le [(m+n)^2\gamma_*]^{\frac{b}{2}-1}[\Gamma(p)]^{-(m+n)}\Gamma(mp)\Gamma(np)(\prod_{i=1}^m x_i\prod_{j=1}^n y_j)^{-1}$$

$$\cdot\int_0^\infty\left(\frac{x_k}{max\{x_1,\cdots,x_m\}}\frac{y_l}{max\{y_1,\cdots,y_n\}}\right)^{pu_3}u_3^{m+n-b-1}du_3$$

$$= [(m+n)^2\gamma_*]^{\frac{b}{2}-1}[\Gamma(p)]^{-n}\Gamma(np)(\prod_{i=1}^m x_i\prod_{j=1}^n y_j)^{-1}$$

$$\cdot\int_0^\infty u_3^{m+n-b-1}e^{-u_3(-plog\frac{x_k}{max\{x_1,\cdots,x_m\}}\cdot\frac{y_l}{max\{y_1,\cdots,y_n\}})}du_3$$

$$= [(m+n)^2\gamma_*]^{\frac{b}{2}-1}[\Gamma(p)]^{-(m+n)}\Gamma(mp)\Gamma(np)(\prod_{i=1}^m x_i\prod_{j=1}^n y_j)^{-1}$$

$$\cdot\Gamma(m+n-b)\left[-plog\frac{x_k}{max\{x_1,\cdots,x_m\}}-plog\frac{y_l}{max\{y_1,\cdots,y_n\}}\right]^{-(m+n-b)}$$

$$< \infty.$$

The remaining cases can be similarly verified. Hence the proof is complete.

Next, we provide the marginal posterior densities of θ_1 under the priors π_J, π_R , and π_M for $b \le 2$ in (3.4). The proofs are omitted.

Theorem 4.2. Under the priors π_J, π_R and π_M for $b \le 2$ in (3.4), the marginal posterior densities of $\theta_1 = (\frac{\eta_2}{\eta_1})^\beta$ are, respectively, given by

$$\pi_J(\theta_1|\underline{x}, \underline{y}) \propto \theta_1^{mp-1} \int_0^\infty u_3^{m+n-1} h(\theta_1, u_3|\underline{x}, \underline{y}) du_3,$$
$$\pi_R(\theta_1|\underline{x}, \underline{y}) \propto \theta_1^{mp-1}[mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^{-1} \int_0^\infty u_3^{m+n-1} h(\theta_1, u_3|\underline{x}, \underline{y}) du_3,$$

and

$$\pi_M(\theta_1|\underline{x}, \underline{y}) \propto \theta_1^{mp-1}[mnp(log\theta_1)^2 + (m+n)^2\gamma_*]^{\frac{b}{2}-1} \int_0^\infty u_3^{m+n-b-1} h(\theta_1, u_3|\underline{x}, \underline{y}) du_3,$$

where

$$h(\theta_1, u_3|\underline{x}, \underline{y}) = (\prod_{i=1}^m x_i \prod_{j=1}^n y_j)^{pu_3} [\theta_1 \sum_{i=1}^m x_i^{u_3} + \sum_{j=1}^n y_j^{u_3}]^{-(m+n)p}.$$

Clearly, the normalizing constant for the marginal posterior densities of θ_1 require two dimensional integration. Once we have the marginal posterior densities of θ_1, we can compute posterior expected values of $R_{r,k}$ under the priors π_J, π_R and π_M for $b \le 2$ in (3.4) . This will not given here.

5. SIMULATION RESULTS

In this section, we investigate how the matching prior π_M for $b = 1$ in (3.4) compare with Jeffreys' prior π_J and the reference prior π_R in finding the estimated values for $R_{r,k}$ in (1,1) when p is known, particularly $p = 1$. In this case, $R_{r,k}$ reduces to

$$R_{r,k} = 1 - \prod_{i=r}^k \frac{i}{\theta_1 + i}, \tag{5.1}$$

where $\theta_1 = \left(\frac{\eta_2}{\eta_1}\right)^\beta$. The results of Section 4 lead easily to inferences about $R_{r,k}$ without explicit knowledge of the posteriors of $R_{r,k}$ due to the fact that $R_{r,k}$ in (5.1) is a monotone increasing function of θ_1.

First, Table 1 gives the simulated values of θ_1 by Gibbs Sampler(100 iterations, 100 samples) using the full posteriors of $(\theta_1, \theta_2, \theta_3)$ in (4.1) under the noninformative priors π_J, π_R, and π_M for $b = 1$ in (3.4) when $m = n = 3$.

Table 1. True values and simulated values of $\theta_1 = (\frac{\eta_2}{\eta_1})^\beta$ by Gibbs sampler under priors π_J, π_R, and π_M for $b = 1$ in (3.4)

β	η_1	η_2	$\theta_1 = (\frac{\eta_2}{\eta_1})^\beta$	θ_1^J	θ_1^R	θ_1^M
0.5	2	3	1.22474	1.47260	1.40029	1.1888
	3	2	0.81649	0.87561	0.84004	0.81705
1	2	3	1.50000	1.84315	1.66416	1.53717
	3	2	0.66667	0.50983	0.67095	0.68487
2	2	3	2.25000	2.30486	2.27416	2.21340
	3	2	0.44435	0.49561	0.46731	0.43330

Next, Table 2 provides the corresponding estimated values of $R_{r,k}$ in (5.1) for $k = 3$ and $r = 1, 2, 3$ substituting the θ_1 values in Table 1 into (5.1).

Table 2. True values and the corresponding estimated values of $R_{r,3}$, $r = 1, 2, 3$, based on the θ_1 values in Table 1

r	β	η_1	η_2	$R_{1,3}$	$R_{1,3}^J$	$R_{1,3}^R$	$R_{1,3}^M$
1	0.5	2	3	0.80204	0.84376	0.83293	0.79479
		3	2	0.69271	0.71296	0.70101	0.69291
	1	2	3	0.84762	0.88662	0.86822	0.85265
		3	2	0.63182	0.54888	0.63378	0.64005
	2	2	3	0.91726	0.92050	0.91871	0.91499
		3	2	0.50659	0.54013	0.52202	0.49892

r	β	η_1	η_2	$R_{2,3}$	$R_{2,3}^J$	$R_{2,3}^R$	$R_{2,3}^M$
2	0.5	2	3	0.55960	0.61369	0.59899	0.55083
		3	2	0.44181	0.46163	0.44984	0.44201
	1	2	3	0.61905	0.67764	0.64892	0.62614
		3	2	0.38637	0.31888	0.38806	0.39354
	2	2	3	0.73109	0.73727	0.73384	0.72685
		3	2	0.28734	0.31222	0.29865	0.28180

r	β	η_1	η_2	$R_{3,3}$	$R_{3,3}^J$	$R_{3,3}^R$	$R_{3,3}^M$
3	0.5	2	3	0.28990	0.32925	0.31823	0.28382
		3	2	0.21394	0.22593	0.21876	0.21405
	1	2	3	0.33333	0.38057	0.35680	0.33880
		3	2	0.18182	0.14526	0.18277	0.18586
	2	2	3	0.42857	0.43448	0.43119	0.42456
		3	2	0.12901	0.14178	0.13478	0.12621

For most of the cases presented in Table 2, we see that the matching prior π_M for $b = 1$ in (3.4) performs better than the Jeffreys' prior in (3.1) and the reference prior π_R in (3.2) in estimating the system reliability $R_{r,k}$ for $k = 3$ and $r = 1, 2, 3$.

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