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Evaluation of Performance Measures for Redundant Systems

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Abstract. This paper considers some reliability indices, such as availability, average availability, reliability and steady state availability, of a redundant system with the function of switchover processing. We also derive the confidence limits for steady state availability of such system. The system, which is considered in this paper, consists of an active unit, a standby unit and a switchover device. In addition, the switchover processing is controlled by a control module. The effect of failure of the control module is taken into account to develop our reliability model for the redundant structure. Numerical examples are presented to illustrate our results.

Key Words : *Redundancy, Availability, Steady-state avalability, Standby redundant system, Transition probability.*

1. INTRODUCTION

Redundancy is defined as the use of additional components or units for satisfactory operation of a system and the standby redundant structures such as electric power generator, UPS and airplane jet engines are widely used to improve the system reliability. In a two-unit repairable standby redundant system, the standby unit

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starts operating immediately upon the failure of the active unit and once the failed active unit is repaired, it assumes the position of standby unit. Thus, these two units alternate their positions either active or standby whenever the failure or repair occurs. Depending on the readiness(or consequently, the failure rate) of standby unit, it is classified as hot, cold or warm standby unit. The cold standby unit does not fail when it remains standby and thus, the failure rate of cold standby unit equals zero. The failure rate of hot standby unit is the same as that of active unit, while the warm standby unit has a smaller failure rate than the active unit, but its failure rate is greater than zero. More details are given in Elsayed (1996). Kumar and Agarwal (1980) also present excellent summaries for the cold redundant structure.

It is common that the performance of the system is analyzed with respect to its reliability characteristics, such as reliability function, availability, MTBF, mean residual life function and so on. One of the most widely used performance criteria of repairable systems is an availability which is defined as the probability that a system is operating satisfactorily when it is required to perform the given mission. Because of the fact that the availability is an important measure to evaluate the performance of the system, many researchers have worked on these subjects quite extensively.

Lim (1996) and Lim and Koh (1997) study a redundant system with the function of switchover processing which consists of three units ; an active unit, a standby unit, a switchover device. Figure 1 shows a reference model for such a redundant system. These articles assume that the failure rate of the system increases by installing the switchover processing, since the failure of the control module can cause the failure of the system. In order to develop a reliability model, they distribute such increment of the failure rate to each unit of the system in such a way that the failure rate of each unit increases by $\lambda_\alpha = \alpha\lambda$, where $0 \leq \alpha \leq 1$ and λ is the failure rate of the unit without the switchover processing. Note that $\alpha = 0$ implies no failure of the control module.

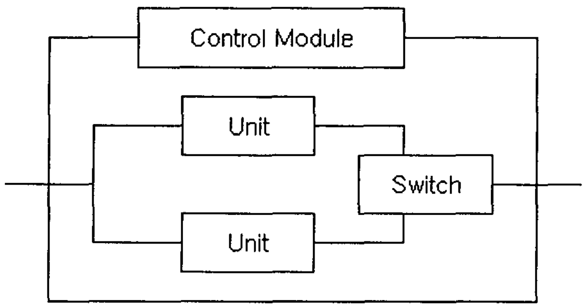


Figure 1. A model of redundant structure with a function of switchover processing.

Chandrasekhar and Natarajan (1994) evaluate the $100(1-\xi)\%$ confidence limits for the steady state availability of two units cold standby and n units parallel system with an additional assumption that an operating unit can also fail while the other unit is in the second stage of repair.

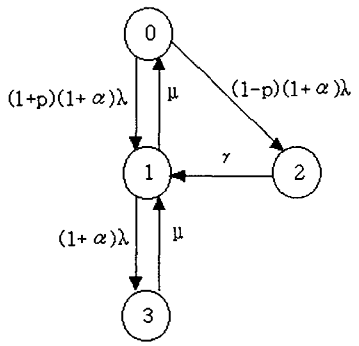
In this paper, we consider a redundant system with switchover processing discussed in Lim (1996) and Lim and Koh (1997). In Section 2, we derive several reliability measures, such as availability, average availability, reliability, steady state availability and $100(1-\xi)\%$ confidence limit for steady state availability. Section 3 presents numerical examples to illustrate our results.

Throughout this paper, we assume the followings:

1) All units are independent and have exponential life distributions, each unit having a mean life of $1/\lambda$, and the repair times of unit and switch are exponentially distributed with means of $1/\mu$ and $1/\gamma$, respectively.
2) The probability of successful switchover operation is assumed to be equal to p.
3) The type of standby unit in the redundant system is hot standby unit.

2. MEASURES FOR HOT STANDBY REDUNDANT SYSTEM

We define four states of the system and draw the state transition diagram(STD) as shown in Figure 2. The states 2 and 3 represent the failure of the system. The state 2,



State	Description
0	Duplex
1	Simplex
2	Uncoverage Outage
3	Failure

Figure 2. State transition diagram of HSRS.

which represents uncoverage outage, is caused by the failure of active unit while the control module is in the failure state and the state 3 is due to the failures of both units.

2.1 Time Dependent Transition Probabilities and Related Measures

Let i, $i = 0, 1, 2, 3$, represent the states of hot standby redundant system(HSRS) and let $p_i(t)$ be the probability that the system is in state i at time t. The flow rate equations can be established by consideration of the fact that the flow rate out of the system must be equal to the flow rate into the state. Thus, we have

$$\begin{aligned}
p_0'(t) &= -2(1+\alpha)\lambda p_0(t) + \mu p_1(t) \\
p_1'(t) &= -(\mu + (1+\alpha)\lambda)p_1(t) + (1+p)(1+\alpha)\lambda p_0(t) + \gamma p_2(t) + \mu p_3(t) \\
p_2'(t) &= -\gamma p_2(t) + (1-p)(1+\alpha)\lambda p_0(t) \\
p_3'(t) &= -\mu p_3(t) + (1+\alpha)\lambda p_1(t),
\end{aligned} \tag{2.1}$$

where $p_i'(t) = dp_i(t)/dt$, $i = 0,1,2,3$.

To solve these equations simultaneously for $p_0(t)$ and $p_1(t)$, we may apply the Laplace and the inverse Laplace transformations. The Laplace transform of a function $f(t)$ is defined as

$$\tilde{f}(s) = \int_0^\infty \exp(-st)f(t)dt.$$

Note that for $i = 0,1,2,3$, the Laplace transform of $p_i'(t)$ is obtained as

$$\tilde{p_i'}(s) = -p_i(0) + s\tilde{p}_i(s).$$

Using the initial conditions, $p_0(0) = 1$ and $p_1(0) = p_2(0) = p_3(0) = 0$, it is straightforward to obtain

$$p_0(t) = N_1\exp(-\beta_1 t) + N_2\exp(-\beta_2 t) + N_3\exp(-\beta_3 t) + N_4 \tag{2.2}$$

and

$$p_1(t) = M_1\exp(-\beta_1 t) + M_2\exp(-\beta_2 t) + M_3\exp(-\beta_3 t) + M_4, \tag{2.3}$$

where $-\beta$'s are the roots of the following third order equation

$$\begin{aligned}
s^3 &+ (2\mu + 3(1+\alpha)\lambda + \gamma)s^2 + \Big(\mu^2 + \mu\lambda(1+\alpha)(3-p) + (2\mu + 3(1+\alpha)\lambda)\gamma \\
&+ 2(1+\alpha^2)\lambda^2\Big)s + \mu^2(\lambda(1+\alpha)(1-p) + \gamma) + 2\lambda\gamma(1+\alpha)(\mu + (1+\alpha)\lambda) \\
&= 0
\end{aligned}$$

and N_j and M_j are

$$N_j = (\gamma - \beta_j)(r_1 - \beta_j)(r_2 - \beta_j)\Big/\prod_{\substack{k=1\\k\neq j}}^{4}(\beta_k - \beta_j)$$

and

$$M_j = \lambda(1+\alpha)(1+p)(2\gamma/(1+p) - \beta_j)(\mu - \beta_j)\Big/\prod_{\substack{k=1\\k\neq j}}^{4}(\beta_k - \beta_j)$$

for $j = 1, 2, 3, 4$ and r_1 and r_2 are $(-((1+\alpha)\lambda + 2\mu) + \sqrt{\lambda(1+\alpha)(\lambda(1+\alpha)+4\mu)})/2$ and $(-((1+\alpha)\lambda + 2\mu) - \sqrt{\lambda(1+\alpha)(\lambda(1+\alpha)+4\mu)})/2$, respectively.

Given the expressions for $p_0(t)$ and $p_1(t)$, we evaluate the following reliability measures for HSRS.

Availability and Average Availability of HSRS

The availability of HSRS at time t, $A(t)$, and average availability of the system in $(0, t]$, $\bar{A}(t)$, are obtained as

$$
\begin{aligned}
A(t) &= p_0(t) + p_1(t) \\
&= \sum_{i=1}^{3}(N_i + M_i)\exp(-\beta_i t) + (N_4 + M_4)
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{A}(t) &= \frac{1}{t}\int_0^t A(u)du \\
&= \sum_{i=1}^{3}(N_i + M_i)(1 - \exp(-\beta_i t))/(\beta_i t) + (N_4 + M_4),
\end{aligned}
$$

respectively. The expected working time of HSRS during the time interval $(0, t]$ can be evaluated by $E_w(t) = \int_0^t [p_0(u) + p_1(u)]du$. It follows that the expected down time of HSRS during $(0, t]$, $E_d(t)$, is $t - E_w(t)$.

Reliability and Mean Time to First Failure of HSRS

By letting μ and γ approach to zero, $A(t)$ can be interpreted as the reliability of HSRS. In this case, the system is considered as a nonrepairable system and its reliability is obtained as

$$R(t) = (1 + p)\exp(-(1+\alpha)\lambda t) - p\exp(-2(1+\alpha)\lambda t)$$

for $t \geq 0$. Thus, the mean time to first failure(MTTFF) of the system is

$$
\begin{aligned}
\text{MTTFF} &= \int_0^\infty R(u)du \\
&= \frac{2+p}{2(1+\alpha)\lambda}.
\end{aligned}
$$

2.2 Steady-State Transition Probabilities and Related Measures

To obtain the steady-state transition probabilities, we replace $p_i(t)$, $i = 0, 1, 2, 3$, by constants p_i(and thus, $p_i^{'}(t)$ by 0) in equations of (2.1) and solve these equations simultaneously for p_0 and p_1, where p_i can be interpreted as a steady-state transition probability of state i. These equations are given as

$$\begin{aligned} \mu p_1 &= 2(1+\alpha)\lambda p_0 \\ (\mu+(1+\alpha)\lambda)p_1 &= (1+p)(1+\alpha)\lambda p_0 + \gamma p_2 + \mu p_3 \\ (1-p)(1+\alpha)\lambda p_0 &= \gamma p_2 \\ (1+\alpha)\lambda p_1 &= \mu p_3. \end{aligned} \tag{2.4}$$

Straightforward calculation yields

$$p_0 = \frac{\mu^2}{\mu^2 + 2\mu(1+\alpha)\lambda + (\mu^2/\gamma)(1-p)(1+\alpha)\lambda + 2(1+\alpha)^2\lambda^2} \tag{2.5}$$

and

$$p_1 = \frac{2\mu(1+\alpha)\lambda}{\mu^2 + 2\mu(1+\alpha)\lambda + (\mu^2/\gamma)(1-p)(1+\alpha)\lambda + 2(1+\alpha)^2\lambda^2}, \tag{2.6}$$

respectively. Therefore, the steady-state availability of the system, denoted by A_H, is equal to $p_0 + p_1$. Although it may not be feasible to prove analytically, the value of $A(t)$ is shown to approach to A_H numerically as t becomes sufficiently large.

The following theorem compares the availabilities of a single component system and a HSRS. It is well known that the availability of a single component, denoted by A_s, is equal to $\mu/(\lambda+\mu)$.

Theorem 1. Let $\delta = 2\lambda + \mu^2/\gamma$. Given that $(-\delta + \sqrt{\delta^2 - 8\lambda\mu((\mu/\gamma)-1)})/4\lambda < \alpha < (-2\lambda + \sqrt{4\lambda^2 + 8\lambda\mu})/4\lambda$, there exist a $p^* \in [0,1]$ such that $A_S \geq A_H$ for $0 \leq p \leq p^*$ and $A_S \leq A_H$ for $p^* \leq p \leq 1$, where $p^* = 1 - (\mu - 2\alpha(1+\alpha)\lambda)\gamma/\mu^2(1+\alpha)$.

Proof. We note that A_H is non-decreasing in p because each of p_0 and p_1 is non-decreasing in p and A_S is a constant. Hence, it is sufficient to show that when $p = 0$, $A_S \geq A_H$ and when $p = 1$, $A_S \leq A_H$. It is somewhat tedious but straightforward to show that when $p = 0$, $A_S \geq A_H$ if $\alpha > (-\delta + \sqrt{\delta^2 - 8\lambda\mu((\mu/\gamma)-1)})/4\lambda$. Similarly, we can show that when $p = 1$, $A_S \leq A_H$ if $\alpha < (-2\lambda + \sqrt{4\lambda^2 + 8\lambda\mu})/4\lambda$. Thus, the existence and uniqueness of p^* is established.

2.3 Confidence Limits for Steady-State Availability of HSRS

To derive the confidence limit for the steady-state availability of HSRS, we consider only the special case when the repair rate of switchover system is proportional to the repair rate of each unit, that is $\gamma = c\mu$ for some positive constant.

Let $X_1, X_2, \cdots, X_n$ be a random sample of times to failure and follow exponential distribution with mean $1/\lambda$, and let $Y_1, Y_2, \cdots, Y_m$ be a random sample of times to repair and follow exponential distribution with mean $1/\mu$. We let $\bar{X}$ and $\bar{Y}$ denote the sample means of times to failure and times to repair, respectively, where

$\bar{X} = \frac{1}{n}\sum_{l=1}^{n} X_l$ and $\bar{Y} = \frac{1}{m}\sum_{l=1}^{m} Y_l$. It is well known that $\bar{X}$ and $\bar{Y}$ are the maximum likelihood estimators(MLE) of $1/\mu$ and $1/\lambda$, respectively. Thus, the MLE of $\theta = \lambda/\mu$ is obtained as $\hat{\theta} = \hat{\lambda}/\hat{\mu}$.

Substituting γ, given in (2.5) and (2.6), with $c\mu$, $c > 0$, and replacing θ by $\hat{\theta}$, the MLE of steady state availability is obtained as

$$\hat{A}_H = \frac{1 + 2(1+\alpha)\hat{\theta}}{1 + 2(1+\alpha)\hat{\theta} + (1-p)(1+\alpha)\hat{\theta}/c + 2(1+\alpha)^2\hat{\theta}^2}.$$

In order to derive the sampling distribution of $\hat{\theta}$, we use the well known fact that $2n\lambda\bar{X}$ and $2m\mu\bar{Y}$ are distributed according to the Chi-square distributions with $2n$ and $2m$ degrees of freedom, respectively. Since the times to failure and the times to repair are independent, it follows that $\bar{X}$ and $\bar{Y}$ are independent. Thus, the random variable F^*, defined by

$$F^* = \frac{2n\lambda\bar{X}}{2n} \Big/ \frac{2m\mu\bar{Y}}{2m} = \frac{\theta}{\hat{\theta}},$$

has a F-distribution with $2n$ and $2m$ degrees of freedom. Let $F_{\xi}(2n, 2m)$ be the ξ-percentile of $F(2n, 2m)$. Then, a $100(1-\xi)\%$ upper confidence limit(UCL) for steady-state availability of the system is constructed as follows:

$$\begin{aligned} 1 - \xi/2 &= P[F^* \geq F_{\xi/2}(2n, 2m)] \\ &= P[A_H \leq \left(1 + 2(1+\alpha)\hat{\theta}F_{\xi/2}(2n,2m)\right) \Big/ \left(1 + 2(1+\alpha)\hat{\theta}F_{\xi/2}(2n,2m) + \right. \\ &\qquad \left. ((1-p)(1+\alpha)\hat{\theta}F_{\xi/2}(2n,2m)/c) + 2(1+\alpha)^2\hat{\theta}^2F_{\xi/2}^2(2n,2m)\right)]. \end{aligned}$$

Note that the second equality holds since A_H is a nonincreasing function of θ. Thus, a $100(1-\xi)\%$ UCL for A_H is given by

$$\begin{aligned} A_{\text{UCL}} &= \left(1 + 2(1+\alpha)\hat{\theta}F_{\xi/2}(2n,2m)\right) \Big/ \left(1 + 2(1+\alpha)\hat{\theta}F_{\xi/2}(2n,2m) + \right. \\ &\qquad \left. ((1-p)(1+\alpha)\hat{\theta}F_{\xi/2}(2n,2m)/c) + 2(1+\alpha)^2\hat{\theta}^2F_{\xi/2}^2(2n,2m)\right). \quad (2.7) \end{aligned}$$

Using the fact that $\xi/2 = P[F^* \geq F_{1-\xi/2}(2n,2m)]$ and $F_{1-\xi/2}(2n,2m) = 1/F_{\xi/2}(2m,2n)$ we can derive a $100(1-\xi)\%$ lower confidence limit(LCL) for A_H as follows:

$$\begin{aligned} A_{\text{LCL}} &= \left(1 + 2(1+\alpha)\hat{\theta}/F_{\xi/2}(2m,2n)\right) \Big/ \left(1 + 2(1+\alpha)\hat{\theta}/F_{\xi/2}(2m,2n) + \right. \\ &\qquad \left. ((1-p)(1+\alpha)\hat{\theta}/(cF_{\xi/2}(2m,2n))) + 2(1+\alpha)^2\hat{\theta}^2/F_{\xi/2}^2(2m,2n)\right) (2.8) \end{aligned}$$

From (2.7) and (2.8), we obtain a two sided $100(1-\xi)\%$ confidence limit for A_H.

3. NUMERICAL EXAMPLES

In this section, we evaluate the values of $p_0(t)$, $p_1(t)$, $A(t)$, $\bar{A}(t)$, $E_w(t)$ and $R(t)$. To estimate the parameters λ, μ and θ, we use the following data for times to failure (Data 1) and times to repair (Data 2), which are obtained under the assumption that both times follow exponential distributions with means of $1/\lambda$ and $1/\mu$, respectively.

Data 1	95.7	98.5	93.4	97.2	100.5
Data 2	6.2	3.8	7.9	7.3	5.4

Based on the data, the MLE of λ, μ and θ are obtained as 0.0103, 0.163399, and 0.063054, respectively.

For Figures 3 and 4 and Table 1, we set $c = 1$ and assume that the values of both p and α are 0.0, 0.3, 0.6, 1.0. Figure 3 presents the graphical behaviors of $A(t)$ for various p when $\alpha = 0.3$ and it shows that the availability increases as the value of p becomes higher. Figure 4 shows that the availability decreases as the value of α increases for $p = 0.9$. Table 1 gives the values of $p_0(t)$, $p_1(t)$, $A(t)$, $\bar{A}(t)$, $E_w(t)$ and $R(t)$ for $\alpha = 0.3$ and $p = 0.9$.

It is easy to obtain for $\alpha = 0.3$ and $p = 0.9$ that the MTTFF is 108.259 and the steady-state availability is 0.981751. Using the formulas (2.7) ans (2.8), a 95% confidence limit for the steady-state availability is obtained as (0.91201, 0.99597).

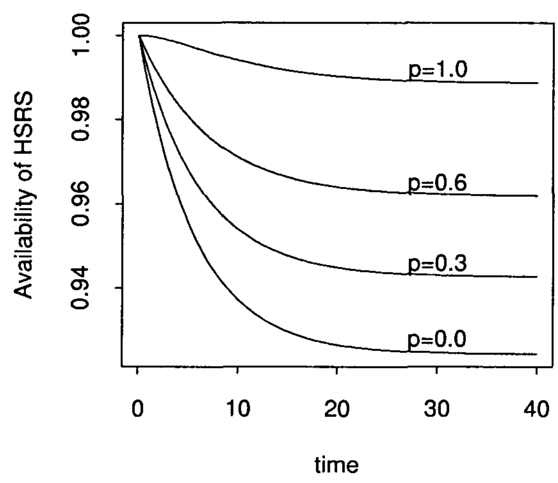


Figure 3. Availability of HSRS, $A(t)$, for various p's with $\alpha = 0.3$.

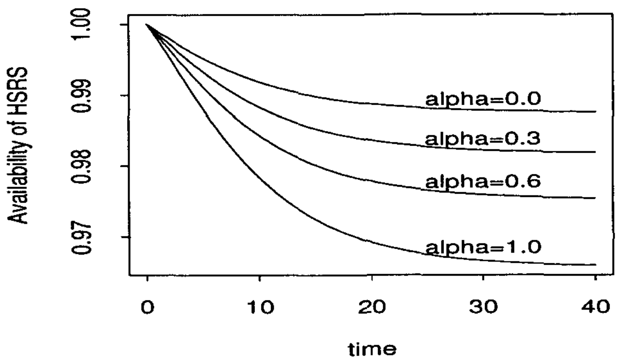


Figure 4. Availability of HSRS, $A(t)$, for various α's with $p = 0.9$.

Table 1. Values of $p_0(t)$, $p_1(t)$, $A(t)$, $\bar{A}(t)$, $E_w(t)$ and $R(t)$ with $\alpha = 0.3$ and $p = 0.9$.

Time(t)	$p_0(t)$	$p_1(t)$	$A(t)$	$\bar{A}(t)$	$E_w(t)$	$R(t)$
3	0.938	0.058	0.996	0.998	2.99	0.995
5	0.911	0.082	0.993	0.997	4.98	0.990
7	0.893	0.098	0.991	0.995	6.97	0.984
10	0.874	0.114	0.988	0.994	9.94	0.973
20	0.850	0.133	0.983	0.990	19.8	0.927
30	0.845	0.137	0.982	0.987	29.6	0.868
40	0.844	0.138	0.982	0.986	39.4	0.804

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