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A Goodness of Fit Approach to Major Lifetesting Problems

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Abstract. Lifetesting problems have been the subject of investigations for over three decades. Most suggested approaches are markedly different from those used in the related but wider goodness of fit problems. In the current investigation, it is demonstrated that a goodness of fit approach is possible in many lifetesting problems and that it results in simpler procedures that are asymptotically equivalent or better than standard ones. They may also have superior finite sample behavior. Several perennial classes are addressed here. The class of increasing failure rate (IFR) and the class of new better than used (NBU) are addressed first. In addition, we provide testing for a newer and practical class of new better than used in convex ordering (NBUC) due to Cao and Wang (1991). Other classes can be developed similarly and this point is illustrated with the classes of new better than used in expectation (NBUE) and harmonic new better than used in expectation (HNBUE).

Key Words : *life distributions, increasing failure rate distributions, new better than used distributions, goodness of fit testing, asymptotic normality, efficiency, power, Monte Carlo methods.*

I. INTRODUCTION

A significant part of lifetesting problems is concerned with testing whether a life distribution belongs to a nonparametric family of aging. Among the most important families of aging are the increasing failure rate (IFR) family, the new better than used

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(NBU) family, the new better than used in convex ordering (NBUC) family, the new better than used in expectation (NBUE) family, and the harmonic new better than used in expectation (HNBUE) family. There is a containment relation between these families as follows:

$$\text{IFR} \subset \text{NBU} \subset \text{WBUC} \subset \text{NBUE} \subset \text{HNBUE}$$

The null distribution for all of the above families is the exponential. Thus, we often encounter testing $H_0 : A$ life distribution is exponential versus $H_1 : A$ life distribution belongs to an aging family. Dealing with this problem seems to have started in the work of Proschan and Pyke (1967) for the IFR class and was followed by many, including, Barlow and Proschan (1969), Bickel and Doksum (1969), and Ahmad (1975). For the NBU class, the work started by Hollander and Proschan (1972), followed by many, including Koul (1977), Kumazawa (1983) and Ahmad (1994), among others. Testing for the NBUE class originated in Hollander and Proschan (1975), while testing for HNBUE began in Klefsjo (1983) and Ahmad et al. (1999).

In contrast to goodness of fit problems, where the test statistic is based on a measure of departure from H_0 that depends on both H_0 and H_1, most tests in lifetesting settings, including those referenced above, do not use the null distribution in devising the test statistics. This resulted in test statistics that are often difficult to work with and require programming to calculate. Alternatively, we demonstrate in the current work that incorporating H_0 into the measure of departure from it can lead to simpler test statistics that are easy to work with, are asymptotically equivalent in distribution to those cited above and may have equal or higher efficiency than the classical procedures. They also may have better finite sample behaviors. This methodology is applied here to testing for the IFR, NBU, NBUC, NBUE, and HNBUE classes, but it applies to others, as well.

Among the testing procedures for the IFR class is the test discussed in Ahmad (1975). This test is based on the following characterization of the IFR class: F is IFR if and only if for any $x, y \geq 0$,

$$\bar{F}^2\left(\tfrac{x+y}{2}\right) \geq \bar{F}(x)\bar{F}(y), \tag{1.1}$$

where $\bar{F} = 1 - F$ is called the survival function (sf). Ahmad (1975) bases his test on the measure of departure from H_0,

$$\Delta^{(1)} = \int_0^\infty \int_0^\infty [\bar{F}^2\left(\tfrac{x+y}{2}\right) - \bar{F}(x)\bar{F}(y)]dF(x)dF(y). \tag{1.2}$$

By plugging the empirical distribution F_n in (1.2), one gets a test statistic that would reject H_0 for large positive values. An equivalent U-statistic form of the test statistic is:

$$U^{(1)} = C_n^{(1)} \sum_{\Omega^{(1)}} I(\min(X_{i_1}, X_{i_2}) > \tfrac{1}{2}(X_{i_3} + X_{i_4})), \tag{1.3}$$

where $C_n^{(1)} = 4/n(n-1)(n-2)(n-3), \sum_{\Omega^{(1)}}$ extends over all $1 \le i_j \le n, j = 1,\ldots,4,$ such that $(i_1, i_2) \ne (i_3, i_4)$ and $i_1 < i_2$ and $i_3 < i_4$. Now, $H_0 : F$ is exponential, thus, if we denote by F_0, this null distribution, we can take in place of (1.2),

$$\delta^{(1)} = \int_0^\infty \int_0^\infty [\bar{F}^2\left(\tfrac{x+y}{2}\right) - \bar{F}(x)\bar{F}(y)] dF_0(x) dF_0(y). \tag{1.4}$$

Since $\delta^{(1)}$ is a scale invariant, we can take $F_0(x) = 1 - e^{-x}, x \ge 0$. In Section 2, we derive an equivalent form to (1.4) and use it to propose a simple test statistic that has the same limiting distribution as that of (1.3) and has the same asymptotic Pitman efficacy but is much simpler to calculate. We offer the empirical critical points and power of the test using simulation that are even better than Ahmad's test.

Next, the most celebrated test for the NBU class is that of Hollander and Proschan (1972). It is based on the definition that F is NBU if and only if

$$\bar{F}(x+y) \le \bar{F}(x)\bar{F}(y), x, y \ge 0. \tag{1.5}$$

They use the measure of departure from H_0 given by:

$$\Delta^{(2)} = \int_0^\infty \int_0^\infty [\bar{F}(x)\bar{F}(y) - \bar{F}(x+y)] dF(x) dF(y) = \tfrac{1}{4} - \int_0^\infty \int_0^\infty \bar{F}(x+y) dF(x) dF(y). \tag{1.6}$$

Their test statistic is

$$U^{(2)} = \tfrac{1}{4} - C_n^{(1)} \sum_{\Omega^{(2)}} I(X_{i_1}, X_{i_2} + X_{i_3}), \tag{1.7}$$

where $C_n^{(2)} = 2/n(n-1)(n-2), \sum_{\Omega^{(2)}}$ extends over all $1 \le i_j \le n, i = 1,2,3$ and $i_1 \ne i_2$ and $i_1 \ne i_3$ and $i_2 < i_3$. Instead of (1.6), we propose

$$\delta^{(2)} = \int_0^\infty \int_0^\infty [\bar{F}(x)\bar{F}(y) - \bar{F}(x+y)] dF_0(x) dF_0(y). \tag{1.8}$$

Again, since $\delta^{(2)}$ is scale invariant, we take $F_0(x) = 1 - e^{-x}, x \ge 0$. In Section 3, we derive an expression of $\delta^{(2)}$ and use it to propose a new test statistic that is asymptotically equivalent to (1.6) but simpler and easier to handle.

A newer class, which is bigger than the NBU class, is the NBUC class, introduced by Cao and Wang (1991). This class is nontrivially larger than the NBU class. A distribution F is said to be NBUC if and only if:

$$\int_x^\infty \bar{F}(y+u) du \le \bar{F}(y) \int_x^\infty \bar{F}(u) du, x, y \ge 0. \tag{1.9}$$

Again, the exponential distribution is the null distribution for this class. To test H_0 against $H_1^{(3)} : F$ is NBUC and not exponential, one can use the following measure of departure from H_0 :

$$\Delta^{(3)} = \int_0^\infty \int_0^\infty [\bar{\nu}(x)\bar{F}(y) - \bar{\nu}(x+y)] dF(x) dF(y), \tag{1.10}$$

where $\bar{v}(x)=\int_x^\infty \bar{F}(u)du$. Based on a random sample $X_1,\ldots,X_n$ from F a test statistic based on plugging the empirical distribution in (1.10) leads to the following equivalent form

$$U^{(3)}=C_n^{(3)}\sum_{\Omega^{(3)}}\{\tfrac{1}{2}(X_{i_1}-X_{i_2})I(X_{i_1}>X_{i_2})-(X_{i_1}-X_{i_2}-X_{i_3})I(X_{i_1}>X_{i_2}+X_{i_3})\}, \tag{1.11}$$

where $C_n^{(3)}=C_n^{(2)}$ and $\sum_{\Omega^{(3)}}=\sum_{\Omega^{(2)}}$. Instead of (1.10), we propose the measure

$$\delta^{(3)}=\int_0^\infty\int_0^\infty[\bar{v}(x)\bar{F}(y)-\bar{v}(x+y)]dF_0(x)dF_0(y). \tag{1.12}$$

In Section 4, we address this testing procedure and show how it performs.

Next, a nonnegative random variable X with distribution F is said to be new better than used in expectation (NBUE) if and only if

$$\bar{\text{v}}(x)\le\bar{F}(x)\mu,\quad x\ge 0,\mu\ge 0. \tag{1.13}$$

where $\bar{\text{v}}(x)=\int_x^\infty\bar{F}(u)du$ and $\mu=E(X)$.

For testing the hypothesis $H_0:F$ is exponential against $H_1^{(4)}:F$ is NBUE and not exponential, we propose the following measure of departure,

$$\delta^{(4)}=\mu^2\int_0^\infty\bar{F}(x)dF_0(x)-\mu\int_0^\infty\bar{\text{v}}(x)dF_0(x) \tag{1.14}$$

and since this measure is scale invariant then without loss of generality we take $\mu=1$ and thus $F_0(x)=1-e^{-x}$. In section 5, we derive an expression for δ and use it to test H_0 against $H_1^{(1)}$. Note that Hollander and Proschan (1975) used the measure:

$$\Delta^{(4)}=\mu^2\int_0^\infty\bar{F}(x)dF(x)-\mu\int_0^\infty\bar{\text{v}}(x)dF(x). \tag{1.15}$$

Finally, a nonnegative random variable X with distribution F is said to be harmonic new better than used in expectation (HNBUE) if and only if

$$\bar{\text{v}}(x)\le\mu e^{-x/\mu},\quad x\ge 0,\mu\ge 0. \tag{1.16}$$

For testing the hypothesis $H_0:F$ is exponential against $H_1^{(5)}:F$ is HNBUE and not exponential, we propose the following measure of departure,

$$\delta^{(5)}=\mu\int_0^\infty e^{-x/\mu}dF_0(x)-\int_0^\infty\bar{\text{v}}(x)dF_0(x). \tag{1.17}$$

Again since this measure is scale invariant then without loss of generality we take $F_0(x)=1-e^{-x}$. In section 6, we derive an expression for $\delta^{(5)}$ and use it to test H_0 against

$H_1^{(5)}$. Once again, the measure $\delta^{(5)}$ corresponds to the measure $\Delta^{(5)}$ depending only on F.

2. TESTING AGAINST IFR ALTERNATIVES

Recall the measure $\delta^{(1)}$ in (1.4). The following lemma is essential in our development.

LEMMA 2.1. Let X_1 and X_2 be two independent variables with the same *df* F, then

$$\delta^{(1)} = 2Ee^{-X_1} - (Ee^{-X_1})^2 - Ee^{-2\min(X_1,X_2)} - E\min(X_1,X_2)e^{-2\min(X_1,X_2)} \qquad (2.1)$$

PROOF. Note that

$$\delta^{(1)} = \int_0^\infty \int_0^\infty \bar{F}^2\left(\tfrac{x+y}{2}\right)e^{-x-y}dxdy - \left(\int_0^\infty \bar{F}(x)e^{-x}dx\right)^2 = I_1 - I_2^2, \text{ say.} \qquad (2.2)$$

Now,

$$I_2 = \int_0^\infty \bar{F}(x)e^{-x}dx = E\int_0^{X_1} e^{-x}dx = 1 - Ee^{-X_1}, \qquad (2.3)$$

where $F(\bar{F})$ is the *df*(sf) of X_1. Next,

$$\begin{aligned} I_1 &= \int_0^\infty \int_0^\infty \bar{F}^2\left(\tfrac{x+y}{2}\right)e^{-(x+y)}dxdy = \int_0^\infty u\bar{F}^2\left(\tfrac{u}{2}\right)e^{-u}du \\ &= 4\int_0^\infty w\bar{F}^2(w)e^{-2w}du = 4E\int_0^{\min(X_1,X_2)} we^{-2w}dw \\ &= E\int_0^{2\min(X_1,X_2)} \theta e^{-\theta}d\theta = 1 - Ee^{-2\min(X_1,X_2)} - 2E\min(X_1,X_2)e^{-2\min(X_1,X_2)}. \end{aligned} \qquad (2.4)$$

The result now follows by simple algebra.

Let $X_1,\ldots,X_n$ denote a random sample from a distribution F. We wish to test $H_0 : F$ is exponential versus $H_1 : F$ is IFR and not exponential. We note that, under $H_0, \delta^{(1)} = 0$, while it is positive under H_1. Thus, we may be testing on its estimate. A direct empirical estimate of $\delta^{(1)}$ is:

$$\begin{aligned} \hat{\delta}_n^{(1)} &= \tfrac{1}{n(n-1)}\sum_{i_1 \neq i_2}\left\{2e^{-X_{i_1}} - e^{-X_{i_1}-X_{i_2}} - e^{-2\min(X_{i1},X_{i2})} - 2\min(X_{i1},X_{i2})e^{-2\min(X_{i1},X_{i2})}\right\} \\ &= \tfrac{1}{n(n-1)}\sum_{i_1 \neq i_2}\varphi^{(1)}(X_{i1},X_{i2}), \text{ say.} \end{aligned} \qquad (2.5)$$

Since the order of the kernel in (2.5) is two while it is four for (1.3), this procedure is simpler to calculate. It also has the same asymptotic properties as (1.3) as we now show.

THEOREM 2.1. As $n \to \infty, (\hat{\delta}_n^{(1)} - \delta^{(1)})$ is asymptotically normal with mean 0 and variance $\sigma_{(1)}^2 / n$, where $\sigma_{(1)}^2$ is as given in (2.9). Under $H_0, \sigma_{0(1)}^2 = 82/25725$.

PROOF. Using standard U-statistics theory, cf. Lee (1989), we need only evaluate the asymptotic variance, which is equal to

$$\sigma^2 = V\{E[\varphi^{(1)}(X_1,X_2)|X_1]+E[\varphi^{(1)}(X_2,X_1)|X_1]\} \tag{2.6}$$

Recall the definition of $\varphi(X_1,X_2)$ in (2.5), thus, it is not difficult to show that

$$\begin{aligned} E[\varphi^{(1)}(X_1,X_2)|X_1] &= 2e^{-X_1} - e^{-X_1}\int_0^\infty e^{-x}dF(x) - \int_0^{X_1} e^{-2x}dF(x) \\ &\quad -e^{-2X_1}\bar{F}(X_1) - 2\int_0^{X_1} xe^{-2x}dF(x) - 2X_1e^{-2X_1}\bar{F}(X_1). \end{aligned} \tag{2.7}$$

Similarly, we have

$$\begin{aligned} E[\varphi^{(1)}(X_2,X_1)|X_1] &= 2\int_0^\infty e^{-x}dF(x) - e^{-X_1}\int_0^\infty e^{-x}dF(x) - \int_0^{X_1} e^{-2x}dF(x) \\ &\quad -e^{-2X_1}\bar{F}(X_1) - 2\int_0^{X_1} xe^{-2x}dF(x) - 2X_1e^{-2X_1}\bar{F}(X_1). \end{aligned} \tag{2.8}$$

Hence,

$$\begin{aligned} \sigma^2 = V\{&2e^{-X_1} + 2\int_0^\infty e^{-x}dF(x) - 2e^{-X_1}\int_0^\infty e^{-x}dF(x) \\ &-2\int_0^{X_1} e^{-2x}dF(x) - 2e^{-2X_1}\bar{F}(X_1) - 4\int_0^{X_1} xe^{-2x}dF(x) \\ &-4X_1e^{-2X_1}\bar{F}(X_1)\} \end{aligned} \tag{2.9}$$

Under H_0,

$$\sigma^2_{0(1)} = V\{\tfrac{1}{9}[9e^{-X_1} - 1 - 8e^{-3X_1} - 24X_1e^{-3X_1}]\} = \tfrac{82}{25725}.$$

Next, we show that $\hat{\delta}^{(1)}_n$ has the same asymptotic Pitman's efficacy as that of $\hat{\Delta}^{(1)}_n$. However, the calculations here are a lot easier. The asymptotic Pitman efficacy of a statistic T_n is defined by: $\left|\frac{\partial}{\partial\theta}ET_n\right|_{\theta\to\theta_0}/\sigma_0$. But $E\hat{\delta}^{(1)}_n = \delta^{(1)}$. Hence, we get

$$\begin{aligned} AE(\hat{\delta}^{(1)}_n) &= \left|\frac{\partial}{\partial\theta}\delta^{(1)}_\theta\right|_{\theta\to\theta_0}/\sigma_{0(1)} \\ &= \left(\tfrac{25725}{82}\right)^{\frac{1}{2}}\left|8\int_0^\infty u\bar{F}'_{\theta_0}(u)e^{-3u}du - \int_0^\infty \bar{F}'_{\theta_0}(u)e^{-u}du\right|, \end{aligned} \tag{2.10}$$

where $\bar{F}'_{\theta_0}(x) = \frac{d}{d\theta}\bar{F}_\theta(x)\Big|_{\theta\to\theta_0}$.

The following three alternatives are customary candidates used in measuring the efficacies of a procedure in testing exponentiality.

(i) The Weibull Distribution:

$$\bar{F}^{(1)}_\theta(t) = e^{-x^\theta}, x \ge 0, \theta \ge 1.$$

(ii) The Linear Failure Rate Distribution:

$$\bar{F}^{(2)}_\theta(t) = e^{-(x+\frac{\theta}{2}x^2)}, x \ge 0, \theta \ge 0.$$

(iii) The Makeham Distribution:

$$\bar{F}_{\theta}^{(3)}(t) = e^{-[x+\theta(x+e^{-x}-1)]}, x \geq 0, \theta \geq 0.$$

The null exponential is attained at $\theta = 1, 0,$ and 0, respectively. Now, $\bar{F}_{\theta_0}^{(1)\prime}(x) = -x \ln x e^{-x}, \bar{F}_{\theta_0}^{(2)\prime}(x) = \frac{-x^2}{2} e^{-x}$ and $\bar{F}_{\theta_0}^{(3)\prime}(x) = e^{-x}(1 - x - e^{-x})$. Thus, plugging these values into (2.10) and doing the calculations, we get the values of $\left|\frac{\partial}{\partial \theta} \delta_{\theta}^{(1)}\right|_{\theta \to \theta_0}$ equal to $\left(-\frac{1}{8} + \frac{1}{4}\ln 2\right), 1/32$ and 1/75, which are exactly the corrected values of Ahmad's test but are much simpler to calculate.

Before closing this section, we also mention here that, based on Monte Carlo methods, the test statistic $\hat{\delta}_n^{(1)}$ has not only simplicity advantages over earlier ones but also has better power, and it converges to the normal fairly quickly. Based on 10,000 replications, we calculated the critical values at $\alpha = .10, .05,$ and $.01$. They are available from the authors. These values converge to those of the limiting normal for sample sizes in excess of 40 in size, which is a very good rate of convergence. Results for the Ahmad (1975) test were slightly worse. We report here, however, the simulated power of the current test at $\alpha = .05$ for sample sizes of 10, 20, and 30, and show the test performs well

Table 1. Power Calculations for Samples from the Weibull

Sample Size/Shape Parameter θ	1.25	1.50	1.75	2.00
10	0.21625	0.27500	0.64250	0.90250
20	0.25563	0.48750	0.85188	0.98438
30	0.28000	0.61313	0.92875	0.99375

Table 2. Power Calculations for Samples from the Linear Failure Rate

Sample Size/Shape Parameter θ	0.25	0.50	0.75	1.00
10	0.95000	0.98875	0.99500	0.99575
20	0.96775	1.00000	1.00000	1.00000
30	0.97875	1.00000	1.00000	1.00000

Table 3. Power Calculations for Samples from the Makeham

Sample Size/Shape Parameter θ	0.25	0.50	0.75	1.00
10	0.30875	0.63000	0.96625	1.00000
20	0.24750	0.71375	0.99650	1.00000
30	0.22375	0.72750	0.99725	1.00000

3. TESTING AGAINST NBU ALTERNATIVES

Recall the measure $\delta^{(2)}$ in (1.8). The following lemma will allow us to develop a test statistic based on $\delta^{(2)}$.

LEMMA 3.1. Let X be a random variable with distribution F. Then

$$\delta^{(2)} = EXe^{-X} - Ee^{-X} + (Ee^{-X})^2. \tag{3.1}$$

PROOF. Note that

$$\delta^{(2)} = \left(\int_0^\infty \bar{F}(x)e^{-x}dx\right)^2 - \int_0^\infty \int_0^\infty \bar{F}(x+y)e^{-(x+y)}dxdy = I_1^2 - I_2 .$$

But

$I_1 = 1 - Ee^{-X}$, and $I_2 = 1 - Ee^{-X} - EXe^{-X}$. The result now follows.

Based on a random sample $X_1,\dots,X_n$ from a distribution F, we wish to test $H_0 : F$ is exponential against $H_2 : F$ is NBU and not exponential. An unbiased estimate of $\delta^{(2)}$ is given by:

$$\hat{\delta}_n^{(2)} = \tfrac{1}{n(n-1)} \sum_{i_1 \neq i_2} (X_{i_1} e^{-X_{i_1}} - e^{-X_{i_1}} + e^{-X_{i_1} - X_{i_2}})$$

$$= \tfrac{1}{n(n-1)} \sum_{i_1 \neq i_2} \varphi^{(2)}(X_{i_1}, X_{i_2}), \text{ say.} \tag{3.2}$$

In order to show asymptotic normality of $\hat{\delta}_n^{(2)}$, we need only to apply the standard theory of U-statistics, cf. Lee (1989), and evaluate the variance

$$\sigma_{(2)}^2 = V\{E[\varphi^{(2)}(X_1, X_2)|X_1] + E[\varphi^{(2)}(X_2, X_1)|X_1]\}. \tag{3.3}$$

But

$$E[\varphi^{(2)}(X_1, X_2)|X_1] = X_1 e^{-X_1} + e^{-X_1}\int_0^\infty e^{-x}dF(x) - e^{-X_1},$$

and

$$E[\varphi^{(2)}(X_2, X_1)|X_1] = \int_0^\infty xe^{-x}dF(x) + e^{-X_1}\int_0^\infty e^{-x}dF(x) - \int_0^\infty e^{-x}dF(x),$$

Under H_0,

$$\sigma_{0(2)}^2 = V\left\{X_1 e^{-X_1} - \tfrac{1}{4}\right\} = \tfrac{5}{432}. \tag{3.4}$$

We have thus proved,

THEOREM 3.1. As $n \to \infty, (\hat{\delta}_n^{(2)} - \varphi^{(2)})$ is asymptotically normal with mean 0 and variance $\sigma_{(2)}^2 / n$ where $\sigma_{(2)}^2$ is as given in (3.3). Under $H_0, \sigma_{0(2)}^2 = 5/432$.

In order to evaluate the asymptotic efficacy of the test, we get that

$$AE(\hat{\delta}_n^{(2)}) = \left(\tfrac{432}{5}\right)^{\frac{1}{2}} \left| \int_0^\infty \bar{F}_{\theta_0}^{\prime}(x)e^{-x}dx - \int_0^\infty x\bar{F}_{\theta_1}^{\prime}(x)e^{-x}dx \right|. \tag{3.5}$$

Now, since for the three alternatives cited in Section 2, the Weibull, the linear failure rate, and the Makeham, $\bar{F}_{\theta_0}^{\prime}(x)$ are equal to $-x\ln xe^{-x}, \frac{-x^2}{2}e^{-x}$ and $e^{-x}(1-x-e^{-x})$, respectively, we get the values of 1.1619, 0.5095, and 0.2582, respectively, which are the values of the Hollander and Proschan (1972) test but a lot easier to obtain.

Finally, in Tables 4 – 6 we give the simulated power of the test.

Table 4. Power Calculations for Samples from the Weibull

Sample Size/Shape Parameter θ	1.25	1.50	1.75	2.00
10	0.18000	0.59313	0.94438	0.99688
20	0.18688	0.81000	.099125	1.00000
30	0.23313	0.86875	0.99375	1.00000

Table 5. Power Calculations for Samples from the Linear Failure Rate

Sample Size/Shape Parameter θ	0.25	0.50	0.75	1.00
10	0.23250	0.28750	0.29625	0.32000
20	0.40750	0.67875	0.77125	0.79125
30	0.60500	0.93625	0.98750	0.98875

Table 6. Power Calculations for Samples from the Makeham

Sample Size/Shape Parameter θ	0.25	0.50	0.75	1.00
10	0.50750	0.99575	0.99975	1.00000
20	0.59125	0.99625	1.00000	1.00000
30	0.74750	0.99875	1.00000	1.00000

4. TESTING AGAINST NBUC ALTERNATIVES

Recall the measure $\delta^{(3)}$ in (1.12). We state and proof the following.

LEMMA 4.1. Let X be a random variable with distribution F. Then

$$\delta^{(3)} = 1 - (EX)(Ee^{-X}) - (Ee^{-X})^2 - EXe^{-X}. \tag{4.1}$$

PROOF. Note that

$$\begin{aligned}\delta^{(3)} &= \int_0^\infty \bar{\nu}(x)e^{-x}dx \int_0^\infty \bar{F}(x)e^{-x}dx - \int_0^\infty \int_0^\infty \bar{\nu}(x+y)e^{-(x+y)}dxdy \\ &= I_1 I_2 - I_3, \text{ say.}\end{aligned} \tag{4.2}$$

Now, since $\overline{v}(x) = E(X-x)I(X>x)$, we easily get that

$$I_1 = E\int_0^X (X-x)e^{-x}dx = EX - Ee^{-X} - 1. \tag{4.3}$$

Similarly,

$$I_2 = 1 - Ee^{-X}, \text{ and} \tag{4.4}$$

$$I_3 = EXe^{-X} + EX + 2Ee^{-X} - 2. \tag{4.5}$$

This last form follows from the fact that

$$\int_0^\infty \int_0^\infty \overline{v}(x+y)e^{-(x+y)}dxdy = \int_0^\infty xe^{-x}\overline{v}(x)dx = E\int_0^X (X-x)xe^{-x}dx.$$

The result follows by simple algebra.

Based on a random sample $X_1,...,X_n$ and using (4.1), we estimate $\delta^{(3)}$ by:

$$\hat{\delta}_n^{(2)} = \frac{1}{n(n-1)}\sum_{i_1 \neq i_2}(1 - X_{i_1}e^{-X_{i_2}} - e^{-(X_{i_1}+X_{i_2})} - X_{i_1}e^{-X_{i_1}}). \tag{4.6}$$

Setting $\varphi^{(3)}(X_{i_1}, X_{i_2}) = 1 - X_{i_1}e^{-X_{i_2}} - e^{-(X_{i_1}+X_{i_2})} - X_{i_1}e^{-X_{i_1}}$, we have that the asymptotic variance is

$$\sigma_{(3)}^2 = V\{E[\varphi^{(3)}(X_1,X_2)|X_1] + E[\varphi^{(3)}(X_2,X_1)|X_1]\}. \tag{4.7}$$

But

$$E[\varphi^{(2)}(X_1,X_2)|X_1] = 1 - X_1(Ee^{-X_1}) - e^{-X_1}(Ee^{-X_1}) - X_1e^{-X_1}, \tag{4.8}$$

and

$$E[\varphi^{(3)}(X_2,X_1)|X_1] = 1 - e^{-X_1}EX_1 - e^{-X_1}(Ee^{-X_1}) - X_1e^{-X_1} \tag{4.9}$$

Under H_0,

$$\sigma_{0(3)}^2 = V\{\tfrac{7}{4} - 2e^{-X_1} - \tfrac{X_1}{2} - X_1e^{-X_1}\} = \tfrac{17}{432}. \tag{4.10}$$

We have thus proved the following:

THEOREM 4.1. As $n \to \infty, (\hat{\delta}_n^{(3)} - \delta^{(3)})$ is asymptotically normal with mean 0 and variance $\sigma_{(3)}^2/n$ where $\sigma_{(3)}^2$ is as given in (4.7). Under $H_0, \sigma_{0(3)}^2 = \frac{17}{432}$.

In order to evaluate the asymptotic efficacy of the above test, we have

$$AE(\hat{\delta}_n^{(3)}) = \left(\tfrac{432}{17}\right)^{\frac{1}{2}}\left\{\tfrac{1}{2}\int_0^\infty \overline{v}_{\theta_0}^{\prime}(x)e^{-x}dx + \tfrac{1}{2}\int_0^\infty \overline{F}_{\theta_1}^{\prime}(x)e^{-x}dx - \int_0^\infty x\overline{v}_{\theta_0}^{\prime}(x)e^{-x}dx\right\}, \tag{4.11}$$

where $\overline{v}_{\theta_0}^{\prime}(x) = \int_x^\infty \overline{F}_{\theta_0}^{\prime}(u)du$. For the three alternatives mentioned in previous sections, we get the asymptotic efficacies of 1.11694 for the Weibull, 0.94519 for the linear failure rate, and 0.14003 for the Makeham. These values, while less than those for the NBU class, as expected since the NBUC class is larger, they are very close to them indicating that the test is quite good. It is even better for the linear failure rate alternative. The simulated power of this test is quite good (in fact, it is at least .97), and are available from the authors.

5. TESTING AGAINST NBUE ALTERNATIVES

Recall the measure in (1.14). The following lemma will allow us to develop a test statistic based on $\delta^{(4)}$.

LEMMA 5.1. Let X be a variable with distribution F. Then

$$\delta^{(4)} = 2 - 2Ee^{-X} - EX \tag{5.1}$$

PROOF. Note that $\delta^{(4)}$ can be written as

$$\delta^{(4)} = E\int_0^\infty I(X > x)e^{-x}dx - E\int_0^\infty (X - x)I(X > x)e^{-x}dx = I_1 - I_2, \text{ say} \tag{5.2}$$

now, $I_1 = 1 - Ee^{-x}$ and $I_2 = EX - Ee^{-x} - 1$, thus the result follows.

Based on a random sample $X_1, ..., X_n$ from a distribution F. We wish to test $H_0 : F$ is exponential against $H_1^{(4)} : F$ is NBUE and not exponential. We note that, under $H_0, \delta^{(1)} = 0$, while it is positive under $H_1^{(4)}$. Thus we may be testing on its estimate. A direct empirical estimate of $\delta^{(1)}$ is:

$$\hat{\delta}_n^{(1)} = 2 - \frac{2}{n}\sum_i \left\{ e^{-X_i} + \frac{X_i}{2} \right\} \tag{5.3}$$

THEOREM 5.1. As $n \to \infty, \sqrt{n}\left(\hat{\delta}_n^{(4)} - \delta^{(4)}\right)$ is asymptotically normal with mean 0 and variance $\sigma_{(4)}^2$ where $\sigma_{(4)}^2$ is as given in (5.4). Under $H_0, \sigma_{0(4)}^2 = 1/3$.

PROOF. It is straightforward, by noting that $\hat{\delta}_n^{(4)}$ is just an average, thus using the central limit theorem the result follows. For the variance $\sigma_{(4)}^2$, we see that

$$\sigma_{(1)}^2 = V\left[2 - 2e^{-x} - X\right]. \tag{5.4}$$

Which under H_0, becomes

$$\sigma_{0(1)}^2 = E\left[4 + 4e^{-2X} + X^2 - 8e^{-x} - 4X + 4Xe^{-X}\right] = 1/3 \tag{5.5}$$

To perform the above test, calculate $\sqrt{3n}\hat{\delta}_n^{(4)}$ and reject if this value exceeds Z_α the standard normal variate.

Carrying out the efficacy calculations for the above three alternatives, namely Weibull, linear failure rate, and Makeham we get 0.9665, 1.299, and 0.5774 respectively.

We compare the above procedure to that of Hollander and Proschan (1975) and Ahmad et al. (1999), the following is obtained

Test	Weibull	LFR	Makeham
Hollander-Proschan	1.2007	0.8660	0.2886
Ahmad et al.	1.2280	0.7490	0.2800
$\hat{\delta}_n^{(1)}$	0.9665	1.2990	0.5774

Which shows that our test falls behind the other two tests in terms of asymptotic efficacy only in the case of the Weibull alternative, however, it has much higher AE for the LFR and Makeham alternatives.

It is also easy to se that the above test is consistent and unbiased. For samples 5(1)25 and using 10000 replications, the upper percentiles of the statistic $\hat{\delta}_n^{(4)}$, and its power for the above three alternatives for %95 percentile are given in the following tables (7 and 8), respectively, we have chosen the exact values of θ as those chosen by Ahmad et al. (1999) to facilitate the comparison.

Table 7. Critical Values of $\hat{\delta}_n^{(4)}$

n	**%90**	**%95**	**%99**
5	0.22568	0.24604	0.27297
6	0.21647	0.23639	0.26687
7	0.20809	0.22933	0.25598
8	0.20169	0.22235	0.24936
9	0.19462	0.21701	0.24575
10	0.18932	0.20968	0.23900
11	0.18353	0.20526	0.23487
12	0.17796	0.20147	0.23339
13	0.17381	0.19562	0.22850
14	0.16847	0.19196	0.22680
15	0.16483	0.18599	0.21907
16	0.15921	0.18310	0.21767
17	0.15558	0.18045	0.21566
18	0.15121	0.17606	0.21002
19	0.14900	0.17332	0.20719
20	0.14404	0.17245	0.20414
21	0.14159	0.16617	0.20119
22	0.13919	0.16428	0.20076
23	0.13527	0.15994	0.19651
24	0.13527	0.15946	0.19281
25	0.13364	0.15765	0.19156

Table 8. Power estimates of $\hat{\delta}_n^{(4)}$

Distribution	Parameter	Sample size		
	θ	10	20	25
Weibull	2	0.242	0.539	0.843
	3	0.624	0.979	1.000
	4	0.862	1.000	1.000
Linear failure rate	2	0.188	0.648	0.961
	3	0.225	0.747	0.989
	4	0.215	0.749	0.990
Makeham	2	0.409	0.808	0.960
	3	0.519	0.937	0.997
	4	0.556	0.972	0.999

Thus we have shown that, based on Monte Carlo methods, the test statistic $\hat{\delta}_n^{(4)}$ has not only simplicity advantages over earlier ones but also has better power especially for sample sizes > 10.

6. TESTING AGAINST HNBUE ALTERNATIVES

Recall the measure $\delta^{(5)}$ in (1.17). The following lemma will allow us to develop a test statistic based on $\delta^{(5)}$.

LEMMA 6.1. Let X be a random variable with distribution F. Then

$$\delta^{(5)} = \frac{3}{2} - Ee^{-X} - EX \tag{6.1}$$

PROOF. Note that

$$\delta^{(5)} = \int_0^\infty e^{-2x}dx - \int_0^\infty \nu(x)e^{-x}dx = \frac{1}{2} - I_1, \text{ say}. \tag{6.2}$$

Now,

$$I = E\int_0^\infty (X-x)I(X>x)e^{-x}dx$$

$$= E\int_0^X (X-x)e^{-x}dx = EX\int_0^X e^{-x}dx - EX\int_0^X xe^{-x}dx$$

and the result now follows.

Based on a random sample $X_1,...,X_n$ from a distribution F. We wish to test $H_0 : F$ is exponential against $H_1^{(2)} : F$ is HNBUE and not exponential. We note that, under $H_0, \delta^{(5)} = 0$, while it is positive under $H_1^{(2)}$. Thus we may be testing on its estimate. A direct empirical estimate of $\delta_n^{(5)}$ is:

$$\hat{\delta}_n^{(5)} = \frac{3}{2} - \frac{1}{n}\sum_i \left\{e^{-X_i} + X_i\right\} \tag{6.3}$$

THEOREM 6.1. As $n \to \infty, \sqrt{n}\left(\hat{\delta}_n^{(5)} - \delta^{(5)}\right)$ is asymptotically normal with mean 0 and variance $\sigma_{(5)}^2$ where $\sigma_{(5)}^2$ is as given in (3.3). Under $H_0, \sigma_{0(5)}^2 = 7/12$.

PROOF. It is straightforward, by noting that $\hat{\delta}_n^{(5)}$ is just an average, thus using the central limit theorem the result follows. For the variance $\sigma_{(2)}^2$, we see that

$$\sigma_{(2)}^2 = V\left[\frac{3}{2} - e^{-X} - X\right]. \tag{6.4}$$

Which under H_0, becomes

$$\sigma_{0(2)}^2 = E\left[\frac{9}{4} + X^2 + e^{-2X} - 3X - 3e^{-X} + 2Xe^{-x}\right] = 7/12. \tag{6.5}$$

To perform the above test, calculate $\sqrt{12n/7}\hat{\delta}_n^{(5)}$ and reject if this value exceeds Z_α the standard normal variate.

Next, the efficacies of $\hat{\delta}_n^{(5)}$ are calculated for the Weibull, Linear failure rate and Makeham alternatives, we get 0.9665, 1.1456, and 0.5455 respectively. Note that Ahmad et al. (1999) and Ahmad (1995) did not calculate the efficacy for the Weibull distribution so we compare our results with theirs for the LFR Makeham only. We get the following,

Test	LFR	Makeham
Ahmad (1995)	0.8660	0.2880
Ahmad et al. (1999)	0.8900	0.4350
$\hat{\delta}_n^{(5)}$	1.1456	0.5455

Which shows that our test outperforms the other two.

It is also easy to see that the above test is consistent and unbiased. For samples 5(1)25 and using 10000 replications, the upper percentiles of the statistic $\hat{\delta}_n^{(5)}$, and its power for the above three alternatives for %95 percentile are given in the following tables

(9 and 10), respectively, we have chosen the exact values of θ as those chosen by Ahmad et al. (1999) to facilitate the comparison.

Table 9. Critical values of $\hat{\delta}_n^{(5)}$

n	%90	%95	%99
5	0.35793	0.40204	0.45447
6	0.33745	0.38230	0.44215
7	0.31295	0.36179	0.43135
8	0.30022	0.34741	0.41798
9	0.28753	0.33486	0.40349
10	0.27509	0.31835	0.38563
11	0.26729	0.31287	0.37997
12	0.25458	0.30266	0.37063
13	0.24880	0.29237	0.35750
14	0.24107	0.28001	0.34823
15	0.22954	0.27406	0.33753
16	0.22429	0.26805	0.34218
17	0.21679	0.25934	0.32803
18	0.21338	0.25562	0.32650
19	0.20794	0.24548	0.31795
20	0.20252	0.24608	0.31304
21	0.19726	0.23804	0.30513
22	0.19339	0.23511	0.29698
23	0.18818	0.22942	0.29622
24	0.18727	0.22398	0.29151
25	0.18558	0.22227	0.28367

Table 10. Power estimates of $\hat{\delta}_n^{(5)}$

Distribution	Parameter	Sample size		
	θ	10	20	25
Weibull	2	0.888	0.922	0.975
	3	0.910	0.954	0.995
	4	0.957	0.984	0.999
Linear failure rate	2	0.205	0.695	0.911
	3	0.391	0.932	0.997
	4	0.578	0.993	1.000
Makeham	2	0.142	0.439	0.650
	3	0.268	0.760	0.931
	4	0.414	0.925	0.993

Thus we have shown that, based on Monte Carlo methods, the test statistic $\delta_n^{(2)}$ has not only simplicity advantages over earlier ones but also has better power especially for sample sizes > 10.

REFERENCES

Ahmad, I.A. (1994). Classes of statistics useful in testing increasing failure rate average and new better than used life distributions. *J. Statist. Plan. Inf.*, **41**, 141-149.

Ahmad, I.A. (1975). A nonparametric test for the monotonicity of a failure rate function. *Comm. Statist.*, **4**, 967-974.

Ahmad, I.A. (1995). Nonparametric testing of classes of life distributions derived from a convex ordering. *Parisankhyan Samikha*, **2**, 13-18.

Ahmad, I.A., Hendi, M. and Al-Nachawati, H. (1999). Testing new better than used classes of life distribution derived from a convex ordering using kernel methods. *J. Nonparam. Statistics,* **11**, 393-411.

Barlow, R.E. (1968). Likelihood ratio tests for restricted families of probability distributions. *Ann. Math. Statist.*, **39**, 547-560.

Barlow, R.E. and Proschan, F. (1969). A note on tests for monotone failure rate based on incomplete data. *Ann. Math. Statist.*, **40**, 595-600.

Bickel, P.J. and Doksum, K. (1969). Tests for monotone failure rate based on normalized spacings. *Ann. Math. Statist.*, **40**, 1216-1235.

Cao, J.H. and Wang, Y.D. (1991). The NBUC and NWUC classes of life distributions. *J. Appl. Probab.*, **28**, 473-479.

Hollander, M. and Proschan, F. (1975). Tests for mean residual life. *Biometrika*, **62**, 585-592.

Hollander, M. and Proschan, F. (1972). Testing whether new is better than used. *Ann. Math. Statist.*, **43**, 1136-1146.

Klefsjo, B. (1982). The HNBUE and HNWUE classes of life distributions. *Naval Res. Logistics Quarterly*, **24**, 331-344.

Koul, H.L. (1977). A new test for new better than used. *Comm. Statist. Theory Meth.*, **7**, 563-573.

Kumazawa, Y. (1983). Testing for new is better than used. *Comm. Stat. Theor. Meth.*, **12**, 311-321.

Lee, A.J. (1989). *U-Statistics*. Mercel-Dekker, New York, NY.

Proschan, F. and Pyke, R. (1967). Tests for monotone failure rate. *Proc.5th Berkeley Symp.*, *3*, 293-312.

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