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Importance Analysis for Capacitated Network Systems

Gi Mun Jung
Department of Naval Architecture and Ocean Engineering
Chosun University, Kwangju 501-759, Korea

Dong Ho Park and Seung Min Lee
Department of Statistics
Hallym University, Chunchon 200-702, Korea

Abstract. A network, where links have different capacities, is considered to be in functioning state if a specified amount of flow can be transmitted through the network. In this paper, we consider the measures of importance of a link in such networks. We define the structural importance and reliability importance of a link when the required amount of flow is given. We also present the performability importance, which can be used to determine which links should be improved first in order to make the greatest improvement in the expected maximum flow of network. Numerical examples are presented as well for illustrative purpose.

Key Words : *link capacity, maximum flow, reliability, performance, link importance.*

1. INTRODUCTION

A network is represented by a probabilistic graph $G(V, E)$, which consists of a set V of nodes and a set E of links. Each link of the network may have different flow capacity. The links either function or fail with known probability, and no flow can be transmitted through a failed link. The network is required to transmit a specified amount of flow from the source node to the terminal node, and the maximum amount of net flow which can be transmitted from the source node to the terminal node is called the *maximum flow* of the network. The network is considered to be in a functioning state if the maximum flow of network is greater than or equal to a specified level. Examples of such networks include a computer communication network which allows only a fixed amount of data exchange among different terminals of various computer centers and a transport system of a large town which limits maximum traffic on various roads. The performance of the network can be measured by network reliability or by the expected maximum flow of network. Network reliability is the probability of transmitting the required amount of flow successfully from the source node to the terminal node. Many algorithms have been suggested to obtain this capacity related reliability. See, for example, Lee and Park (2001a) and references therein. As a measure of performance of network, Aggarwal (1985) suggests the performance index, which is a normalized expectation of maximum flow of network.

With an appropriate measure being defined for network performance, link importance represents the degree of criticality of the link to the corresponding performance measure of the network. For instance, the reliability importance of a link can be defined as the rate at which the network reliability changes due to the changes in the reliability of that link, and it can be used to determine which links should be improved first in order to make the greatest improvement in network reliability. Birnbaum (1969) discusses measures of component importance in binary systems where each component and the system can be in one of only two states, functioning or failed, and these measures are extended to multistate systems by a number of authors. Kim and Baxter (1987) suggest the more general case where the domain and range of the structure function are both continua. See also Baxter and Lee (1989). Hong and Lie (1993) introduce joint reliability importance of two edges in an undirected network, where each edge and the network assume binary states. Page and Perry (1994) examine the use of reliability polynomials to rank the edges in a graph in terms of overall importance to the graph reliability. Recently, Armstrong (1997) discusses the relation of various importance measures and extends such measures to cover reliability models in which each component has two failure-modes.

In this paper, we consider importance measures of a link in capacitated networks. In Section 2, we describe network reliability when the required amount of flow, *success level* say, is given. Network performability is defined as the expected maximum flow of network. Then, we define the structural importance, reliability importance and performability importance for links in the network. In Section 3, numerical examples for various networks are given for illustrative purpose.

Notations

c_i	flow capacity of link i
$\hat{c}$	capacity vector : $\hat{c} = (c_1, \cdots, c_n)$
p_i	probability that link i is functioning
$\hat{p}$	link reliability vector : $\hat{p} = (p_1, \cdots, p_n)$
X_i	random variable indicating the state of link i
$\hat{X}$	random state vector : $\hat{X} = (X_1, \cdots, X_n)$
$\hat{x}$	binary vector values that $\hat{X}$ can assume
$R^l(\hat{p})$	probability that network is working in l-level
$F(\hat{p})$	performability of network

SI_i^l l-level structural importance for link i
RI_i^l l-level reliability importance for link i
FI_i performability importance for link i

2. MEASURES OF LINK IMPORTANCE

Assumptions

1. The nodes are perfect and each has no capacity limit.
2. The links are independent and either function or fail with known probability.
3. All the links are directed and each link flow is bounded by the capacity of the link.
4. No information or flow can be transmitted through a failed link.

Suppose that the link capacity vector $\hat{c}$ is given, and let $M(\hat{x})$ denote the maximum flow that the network can transmit at state vector $\hat{x}$ of links. We note that $M(\hat{x})$ is non-decreasing in each argument. The methods for computing $M(\hat{x})$ efficiently are discussed extensively in the literature, for example Rai and Soh (1991), Kyandoghere (1998) and Lee and Park (2001b). The network is considered to be in a functioning state if it can transmit a maximum flow which is greater than or equal to a specified success level, l say, i.e. if $M(\hat{x}) \geq l$. Suppose that a success level l is pre-specified. Then the network reliability $R^l(\hat{p})$ can be expressed as

$$R^l(\hat{p}) = P\{M(\hat{X}) \geq l\},$$

where $\hat{p} = (p_1, \cdots, p_n)$ and p_i is the probability that link i is functioning. The network reliability is the probability that the required amount of flow l can be transmitted from the source node to the terminal node, which depends on $\hat{p}$. Now, we first consider the *structural importance* of a link. The structural importance of link i defined by

$$SI_i^l = \frac{1}{2^{n-1}} \sum [S^l(1_i, \hat{x}) - S^l(0_i, \hat{x})],$$

where $S^l(\hat{x})$ is the binary structure function whose value is 1 if $M(\hat{x}) \geq l$, and 0 otherwise. The structural importance SI_i^l is the relative proportion of 2^{n-1} vectors at which the functioning of link i makes the network function, whereas the failure of link i cause the failure of the network. Thus, it represents the degree of criticality of link i to the success of the network in structural sense.

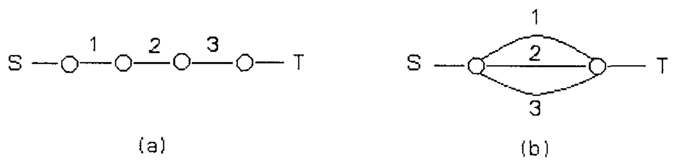

Figure 1. Series and parallel structure

We consider the series and parallel structure shown in Figure 1, each of which is constructed with links 1, 2 and 3 having capacities of $(3, 2, 1)$. The structural importance of the links, varying the levels, for each structure are summarized in Table 1 below.

Table 1. Structural importance for series and parallel structure

Link	SI_i^l for series structure	SI_i^l for parallel structure					
(i)	$l = 1$	$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 5$	$l = 6$
1	0.25	0.25	0.50	0.75	0.75	0.50	0.25
2	0.25	0.25	0.50	0.25	0.25	0.50	0.25
3	0.25	0.25	0	0.25	0.25	0	0.25

Next, we consider the *reliability importance* of a link. The reliability importance of link i is defined by

$$RI_i^l = \frac{\partial R^l(\hat{p})}{\partial p_i}.$$

The reliability importance, RI_i^l, represents the degree of criticality of link i to network reliability, and it can be used to determine which link should be improved first in order to make the greatest improvement of the network with respect to its reliability. We note that $RI_i^l = SI_i^l$ for all l if the links are independent with $p_i = 0.5$, $i = 1, 2, \cdots, n$.

Example 2.1. We consider the series and parallel structure shown in Figure 1. Let the link capacity vector be given as $\hat{c} = (3, 2, 1)$ and let the reliabilities of links be equal, i.e. $p_1 = p_2 = p_3 = p$. The reliability importance of each link for those structures is summarized in Table 2. For the series structure (a), each link has the same reliability importance. Unlike the binary case, the reliability importance of each link in the parallel structure (b) is not the same. For example, when $l = 3$, the reliability of link 1 needs to be improved first in order to make the greatest improvement in reliability of the network. For $l = 2$ or $l = 5$, link 3 is of no importance, since the change in reliability of link 3 has no effect on network reliability. We observe that the reliability importance coincides with the structural importance if $p_i = 0.5$, $i = 1, 2, \cdots, n$.

Table 2. Reliability importance for series and parallel structure

Link	RI_i^l for series structure	RI_i^l for parallel structure					
(i)	$l=1$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$
1	p^2	$1-2p+p^2$	$1-p$	$1-p^2$	$2p-p^2$	p	p^2
2	p^2	$1-2p+p^2$	$1-p$	$p-p^2$	$p-p^2$	p	p^2
3	p^2	$1-2p+p^2$	0	$p-p^2$	$p-p^2$	0	p^2

Now, as a measure of over-all performance of network, we define the network *performability* $F(\hat{p})$ as

$$\begin{aligned} F(\hat{p}) &= E[(M(\hat{X})] \\ &= \sum_l l \cdot [R^l(\hat{p}) - R^{l+1}(\hat{p})], \end{aligned}$$

which is the expected amount of maximum flow of network. Then, based on the concept of network performability, we may evaluate the *perfomability importance* of a link. The performability importance of link i is defined by

$$FI_i = \frac{\partial F(\hat{p})}{\partial p_i}.$$

The performability importance, FI_i, represents the degree of criticality of link i to the over-all performance of network, and it can be used to determine which link should be improved first in order to make the greatest improvement of the network with respect to the expected maximum flow. For the series structure in Example 2.1, $F(\hat{p}) = p_1p_2p_3$ and $FI_1 = p_2p_3$, $FI_2 = p_1p_3$, and $FI_3 = p_1p_2$. Thus, the link with lowest reliability is the most important to the network in performability. For the parallel structure, we have $F(\hat{p}) = 3p_1 + 2p_2 + p_3$ and $FI_1 = 3$. $FI_2 = 2$, and $FI_3 = 1$. Thus, the link with largest capacity is the most important to the network in performability. That is, to increase the reliability of the link with largest capacity would yield the greatest improvement in network performability.

3. NUMERICAL EXAMPLES

In this section, we present some numerical results of link importance for various structures other than series or parallel structures.

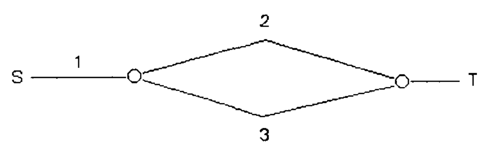

Figure 2. Simple network

Example 3.1. We consider the simple network, given in Figure 2, with the flow capacities $c_1 = 3$, $c_2 = 2$ and $c_3 = 1$. For this particular case, the network reliability in the level $l(= 1, 2, 3)$ can be obtained as follows : $R^1(\hat{p}) = p_1p_2 + p_1p_3 - p_1p_2p_3$, $R^2(\hat{p}) = p_1p_2$ and $R^3(\hat{p}) = p_1p_2p_3$. Let $p_1 = p_2 = p_3 = p = 0.9$. Then, we have $R^1(\hat{p}) = 0.891, R^2(\hat{p}) = 0.81$ and $R^3(\hat{p}) = 0.729$. The performability, $F(\hat{p})$, for the simple network is 2.43. Table 3 gives the values of RI_i^l and FI_i when $\hat{c} = (3, 2, 1)$.

Table 3. Link importance of the simple network

Link Number	Reliability Importance RI_i^l			Performability Importance
(i)	$k = 1$	$k = 2$	$k = 3$	FI_i
1	$2p - p^2$	p	p^2	$3p$
2	$p - p^2$	p	p^2	$2p$
3	$p - p^2$	0	p^2	p

For $l = 1$, the reliability importance of link 1 is the largest and thus, the reliability of link 1 should be improved first in order to make the greatest improvement in reliability of the network. Since the performability importance of link 1 is also the largest among those of three links, reliability of link 1 should be improved first to achieve the greatest improvement in performablity of the network.

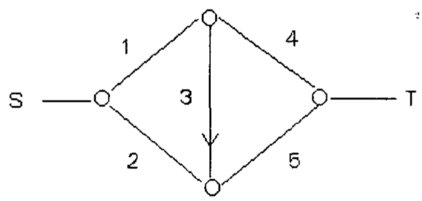

Figure 3. Bridge network

Example 3.2. We consider the bridge network of Figure 3. The flow capacities and link reliabilities are fixed at $\hat{c} = (6, 2, 2, 3, 2)$ and $p_i = p$, $i = 1, 2, \cdots, 5$. Table 4 shows the values of reliability and performability importance of each link for the bridge structure.

Table 4. Link importance of the bridge network

	l	Link $i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
RI_i^l	1	$p + p^2 - 3p^3 + p^4$	$p - 2p^3 + p^4$	$p^2 - 2p^3 + p^4$	$p - 2p^3 + p^4$	$p + p^2 - 3p^3 + p^4$
	2	$p + p^2 - 3p^3 + p^4$	$p - 2p^3 + p^4$	$p^2 - 2p^3 + p^4$	$p - 2p^3 + p^4$	$p + p^2 - 3p^3 + p^4$
	3	p	0	0	p	0
	4	$2p^3 - p^4$	$p^3 - p^4$	$p^3 - p^4$	$2p^3 - p^4$	$2p^3 - p^4$
	5	$2p^3 - p^4$	$p^3 - p^4$	$p^3 - p^4$	$2p^3 - p^4$	$2p^3 - p^4$
FI_i		$3p + 2p^2 - 2p^3$	$2p - 2p^3$	$2p^2 - 2p^3$	$3p$	$2p + 2p^2 - 2p^3$

For $l = 1$, the reliability of link 1 or link 5 should be improved first in order to make the greatest improvement in reliability of the network. We also observe that $FI_1 > FI_4 > FI_5 > FI_2 > FI_3$. This implies that link 1 is the most important link to improve the performability of the network. Figure 4 presents a graphical comparison of the performability importance of the links.

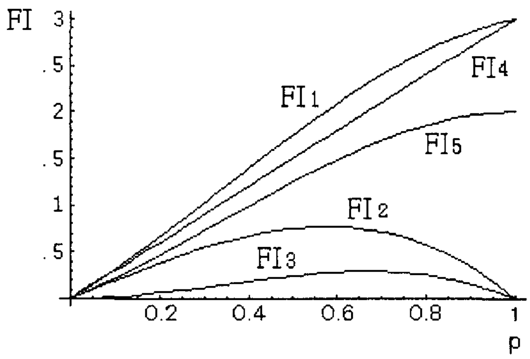

Figure 4. Performability importance of the bridge network

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