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Invited Paper

Analyzing Survival Data by Proportional Reversed Hazard Model

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Abstract. The purpose of this paper is to introduce a proportional reversed hazard rate model, in contrast to the celebrated proportional hazard model, and study some of its structural properties. Some criteria of ageing are presented and the inheritance of the ageing notions (of the base line distribution) by the proposed model are studied. Two important data sets are analyzed: one uncensored and the other having some censored observations. In both cases, the confidence bands for the failure rate and survival function are investigated. In one case the failure rate is bathtub shaped and in the other it is upside bath tub shaped and thus the failure rates are non-monotonic even though the baseline failure rate is monotonic. In addition, the estimates of the turning points of the failure rates are provided.

Key Words : *Survival function, hazard rate, reversed hazard rate, order relations, confidence bands, turning point.*

1. INTRODUCTION

Cox (1972) proportional hazard model (PHM) for survival analysis has been applied extensively to model failure time data. For example, in the comparison of two survival functions $\overline{F}_1(t)$ and $\overline{F}_2(t)$ in a clinical trial, the PHM assumes that $\overline{F}_1(t) = [\overline{F}_2(t)]^\theta$ for some constant θ. In this setting, the parameter θ has the interpretation of relative risk and has intuitive appeal as a descriptive measure in survival analysis. The model also implies that the two hazard rates corresponding to the distribution F_1 and F_2 are proportional and thus have the same monotonic properties.

As an alternate, Gupta et al (1998) proposed a model $F^*(t) = [F(t)]^\theta$, where $F(t)$ is the base line distribution function, and studied the monotonicity of the failure rates in the case of exponentiated Weibull, exponentiated exponential, exponentiated Pareto and exponentiated Gamma family of distributions. This model is flexible enough to accommodate both monotonic as well as non-monotonic failure rates even though the base line failure rate is monotonic. The non-monotonic failure rate models are frequently encountered in modeling failure data, for example, lognormal, inverse Gaussian, mixture inverse Gaussian, see Gupta et al. (1997), Gupta and Akman (1995, 1997). For more applications of non-monotonic failure rates, see Mudholkar et al (1995) and the references therein. Recently Mudholkar and Srivastava (1993), Mudholkar et al (1995) and Mudholkar and Hutson (1996) have modeled various failure time data sets with Weibull exponentiated distribution.

In this paper we further study this model in terms of the reversed hazard rates and review some results related to the ageing classes of distributions in reliability. In addition, we study the Weibull exponentiated family and analyze two important data sets from a reliability point of view. The organization of this paper is as follows: In section 2, we present a motivation of this model in cancer research and study some structural properties of the model. In section 3, a set of sufficient conditions are provided for F^* to inherit the ageing notions of F. Section 4 contains some comparisons of the two distributions F and F^* when θ is treated as random. The results are analogus to those of Frailty models in survival analysis. In section 5, we study the Weibull exponentiated family and examine some structural properties of the model including the shape of the probability density function (pdf) and the failure rate. Section 6 deals with the analysis of Aarset (1987) data and we obtain the maximum likelihood estimates (MLE's) and confidence bands for the failure rate and survival function together with the estimate of the turning point of the failure rate. The goodness of fit of the model is examined by likelihood ratio statistic and Wald statistic. Similar analysis is carried out, in section 7, for a censored data set of Efron (1988).

2. THE MODEL

Let T be a non negative random variable denoting the life length of a component having distribution function $F(t)$ with $F(0)=0$ and pdf $f(t)$. Then the failure rate of T is given by $r(t)=f(t)/\bar{F}(t)$, where $\bar{F}(t)=1-F(t)$ is called the survival (reliability) function of T.

Let T^* be a non-negative random variable such that its distribution function $F^*(t)$ is an exponentiated function of $F(t)$, i.e.

$$F^*(t)=[F(t)]^\theta, t>0, \theta>0. \tag{2.1}$$

The pdf and the failure rate of T^* are given by

$$f^*(t)=\theta F^{\theta-1}(t)f(t) \tag{2.2}$$

and

$$r^*(t)=\theta r(t)g(t), \tag{2.3}$$

where

$$g(t)=\frac{F^{\theta-1}(t)[1-F(t)]}{1-F^\theta(t)}.$$

Note that in this case $g(t)$ is independent of t and hence the hazard rates $r(t)$ and $r^*(t)$ are not proportional. However, the reversed hazard rates $\tau(t)$ and $\tau^*(t)$ (defined below) are proportional.

2.1 Reversed Hazard Rate

To fix the ideas let $a = \inf\{t : F(t) > 0\}$ and $b = \sup\{t : F(t) < 1\}$. Then the reversed hazard rate of T is defined for $t > a$ as

$$\tau(t) = \frac{d}{dt}\ln F(t) = \frac{f(t)}{F(t)} \tag{2.4}$$

Denoting the reversed hazard rate of $F^*(t)$ by $\tau^*(t)$, it is clear that the model is equivalent to saying that $\tau(t)$ and $\tau^*(t)$ are proportional. The reversed hazard rate can be interpreted as follows: Suppose the life time of a unit has reversed hazard rate $\tau(t)$. Then $\tau(t)dt$ is the conditional probability that the unit failed in an infinitesimal interval of width dt preceeding t, given that it failed at or before t. In forensic science and in acturial science, the time elapsed since failure is a quantity of interest in order to predict the exact time of failure. In this case $\tau(t)dt$ provides the probability of failing in $(t-dt, t)$ when a unit is found failed at time t. For more applications of reversed hazard rate, see Gupta and Nanda (2001), Eeckhoudt and Gollier (1995) and Kijima and Ohinishi (1999). In general, the reversed hazard rate has been found to be useful in estimating survival function for left censored data, see Kalbfliesch and Lawless (1989). Block et al (1998), Sengupta and Nanda (1999) and Chandra and Roy (2001) have presented several interesting results regarding the reversed hazard rates, see Kijima (1998) for some applications of reversed hazard rate in the study of continuous time Markov Chains.

We now present an application of the model.

2.2 An Application and Motivation

Tsodikov et al (1997) describe a stochastic model of spontaneous carcinogenesis which allows for a simple pattern of tumor growth kinetics. It is assumed that a tumor becomes detectable when its size attains some threshold level. The tumor growth can be assumed to obey the postulates of birth and death process with two absorbing states so that the first passage time with respect to the upper barrier will correspond to the time of tumor progression. The basic assumptions of the model are as follows:

(1) The initiation event in the process of carcinogenesis is the formation of certain cells. The formation of these cells occur at random times and their sequence in time is modeled as a homogeneous Poisson process.

(2) Once a first malignant cell arises as a result of tumor promotion, its subsequent growth is irreversible and the progression stage begins. It is this clonogenic cell that gives rise to a detectable tumor after a lapse of time, which is thought of as a random variable with cumulative distribution function $F(t)$.

(3) A tumor becomes detectable when its size attains some threshold value N. A linear pure birth process with absorbing barrier N is used to model the dynamics of the tumor growth.

The critical number of tumor cells is represented by $N = cV$, where V is the volume of a tumor and c is the concentration of tumor cells per unit volume. The conditional progression time distribution given the threshold volume v is given by

$$F(t \mid v) = (1 - e^{-\lambda t})^{cv},$$

where λ is the birth rate.
The above is an example of our model where the role of F is played by an exponential distribution and the role of θ is played by cv.

3. SOME CRITERIA OF AGEING

Let T be a continuous positive random variable representing the life of a component. Let F be the distribution function of T and $\overline{F}(t) = 1 - F(t)$ be the reliability or the survival function of T. Then $\overline{F}_t(x) = P(T > x + t \mid T > t)$ is the survival function of a unit of age t. Evidently, any study of the phenomenon of aging should be based on $\overline{F}_t(x)$ and functions related to it. Thus

(1) F is said to be PF_2 (or increasing likelihood ratio property) if $\ln f(t)$ is concave, where $f(.)$ is the density corresponding to $F(.)$ i.e. $f(x+t)/f(x)$ is decreasing in x for all $t > 0$.

(2) F is said to be increasing (decreasing) failure rate if $\overline{F}(x) = \overline{F}(x+t)/\overline{F}(t)$ is decreasing (increasing) in t. If F is absolutely continuous with density f, then F is in $IFR(DFR)$ class if the failure rate $r_F(t) = f(t)/\overline{F}(t)$ is increasing (decreasing).

(3) F is said to have increasing (decreasing) failure rate average, *IFRA*(*DFRA*) if $\int_0^t r_F(x)dx/t$ is increasing (decreasing).

(4) F is said to have new better (worse) than used, $NBU(NWU)$ if $\overline{F}_t(x) \le (\ge) \overline{F}(x)$ for $x \ge 0, t \ge 0$.

(5) F is said to have decreasing (increasing) mean residual life function, $DMRL(IMRL)$ if the mean residual life $\mu_F(t) = \int_t^\infty \overline{F}(x)dx / \overline{F}(t)$ is decreasing (increasing), assuming that the mean $\mu_F(0)$ exists.

(6) F is said to have new better (worse) than used in expectation, $NBUE(NWUE)$ if $\mu_F(t) \le (\ge) \mu_F(0)$ for all $t \ge 0$.

The chain of implications between these classes of distributions is

$$PF_2 \Rightarrow IFR \Rightarrow IFRA \Rightarrow NBU \Rightarrow NBUE$$
$$IFR \Rightarrow DMRL \Rightarrow NBUE.$$

The reverse implications are not true, for counter examples, see Bryson and Siddiqui (1969).

3.1 Inheritance of Aging Notation by F^*

In the following, we present sufficient conditions for F^* to inherit the ageing notion of F.

Theorem 3.1

(1) If $\theta > (<) 1$ *and* $F \in IFR(DFR)$, *then* $F^* \in IFR(DFR)$.

(2) If $\theta > (<) 1$ *and* $F \in NBU(NWU)$, *then* $F^* \in NBU(NWU)$.

(3) If $\theta > (<) 1$ *and* $F \in PF_2$, *then* $F^* \in PF_2$.

For proof, see Gupta et al (1998) for part (a) and Crescenzo (2000) for parts (b) and (c).

Without the condition $F \in IFR(DFR, NBU, NWU, PF_2)$ the following result is true.

Theorem 3.2

(a) If $\theta > 1$ *and* $\frac{d^2}{dt^2} \ln F(t) < \frac{1}{(1-\theta)} \frac{d^2}{dt^2} \ln f(t)$ *for all* $t > 0$, *then* $F^* \in IFR$.

(b) If $\theta < 1$ *and* $\frac{d^2}{dt^2}\ln F(t) > \frac{1}{(1-\theta)}\frac{d^2}{dt^2}\ln f(t)$ *for all* $t > 0$, *then* $F^* \in IFR$.

For proof see Gupta et al (1998).

Remark: Similar conditions can be obtained for F^* to belong to DFR.

3.2 Some Ordered Relations

In this section, we shall present some order relations between F and F^*.

Let X and Y be nonnegative absolutely continuous random variables with distribution functions $F_X(t)$ and $F_Y(t)$ and density functions $f_X(t)$ and $f_Y(t)$, respectively. Then X is said to be smaller than Y in the

(1) likelihood ratio order $(X \leq_{lr} Y)$ if $f_X(t)/f_Y(t)$ decreases in $t > 0$.

(2) hazard rate order $(X \leq_{hr} Y)$ if $r_X(t) \geq r_Y(t)$ for all $t > 0$.

(3) reversed hazard rate order $(X \leq_{rh} Y)$ if $\tau_X(t) \leq \tau\ (t)$, where $\tau_X(t)$ and $\tau_Y(t)$ are the reversed hazard rates of X and Y.

(4) stochastic order $(X \leq_{st} Y)$ if $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for all $t > 0$.

(5) mean residual life order $(X \leq_{MRL} Y)$ if $\mu_X(t) \leq \mu_Y(t)$ for all $t > 0$.

The following implications are well known

$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y$

and

$X \leq_{lr} Y \Rightarrow X \leq_{rh} Y \Rightarrow X \leq_{st} Y.$

Also

$X \leq_{hr} Y \Rightarrow X \leq_{MRL} Y.$

The following theorem compares the distributions F and F^* with respect to the above orderings.

Theorem 3.3

(1) If $\theta > (<)1$, *then* $T \leq_{lr} (\geq_{lr}) T^*$.

(2) If $\theta > (<)1$, *then* $T \leq_{hr} (\geq_{hr}) T^*$.

(3) If $\theta > (<)1$, *then* $T \leq_{MRL} (\geq_{MRL}) T^*$.

(4) If $\theta > (<)1$, *then* $T \leq_{st} (\geq_{st}) \theta T$.

For proof, see Gupta et al (1998) and Crescenzo (2000).

4. COMPARISON OF F AND F^* WHEN Θ IS RANDOM

If Θ is considered as random, our model can be written as

$$F^*(t \mid \Theta = \theta) = [F(t)]^{\theta} .$$

This gives the unconditional distribution function as

$$F^*(t) = \int_0^{\infty} [F(t)]^{\theta} g(\theta) d\theta,$$

where $g(\theta)$ is the probability density function (pdf) of Θ.
Denoting by $\tau(t)$, the baseline reversed hazard rate, the unconditional distribution function can be written as

$$F^*(t) = \int_0^{\infty} e^{\theta T(t)} g(\theta) d\theta, \tag{4.1}$$

where $T(t) = \int_0^t \tau(x) dx$ is the integrated reversed hazard rate. Thus

$$F^*(t) = M_{\Theta}(T(t)),$$

where $M_{\Theta}(s)$ is the moment generating function of Θ at the point s.

Suppose now that the reversed hazard rate of T given $\Theta = \theta$ is $\tau(t \mid \theta)$. In order to find the unconditional reversed hazard rate of T, we proceed as follows:

$$\begin{aligned} \tau^*(t) = \frac{f^*(t)}{F^*(t)} &= \frac{\int_0^{\infty} f(t \mid \theta) g(\theta) d\theta}{F^*(t)} \\ &= \int_0^{\infty} \frac{f(t \mid \theta)}{F(t \mid \theta)} \frac{F(t \mid \theta)}{F^*(t)} g(\theta) d\theta \\ &= \int_0^{\infty} \tau(t \mid \theta) \frac{F(t \mid \theta)}{F^*(t)} g(\theta) d\theta \end{aligned} \tag{4.2}$$

Now the distribution function of Θ given $T < t$ is given by

$$P(\Theta \le \theta \mid T < t) = \int_0^\theta \frac{P(T < t \mid \Theta = \theta)}{P(T < t)} g(\theta) d\theta .$$

Thus the pdf of Θ given $T < t$ is

$$\psi(\theta \mid T < t) = \frac{F(t \mid \theta)}{F^*(t)} g(\theta) \tag{4.3}$$

Hence,

$$\begin{aligned} \tau^*(t) &= \int_0^\infty \tau(t \mid \theta) \psi(\theta \mid T < t) d\theta \\ &= E_{\Theta \mid T<t}(\tau(t \mid \theta) \\ &= E_{\Theta \mid T<t}(\Theta \tau(t)) \\ &= \tau(t) E(\Theta \mid T < t), \end{aligned} \tag{4.4}$$

where $\tau(t)$ is the baseline reversed hazard rate.
Thus,

$$\frac{\tau^*(t)}{\tau(t)} = E(\Theta \mid T < t) \tag{4.5}$$

We shall now show that $E(\Theta \mid T < t)$ is an increasing function of t.
Now,

$$\begin{aligned} E(\Theta \mid T < t) &= \frac{\int_0^\infty \theta F(t \mid \theta) g(\theta) d\theta}{F^*(t)} \\ &= \frac{\int_0^\infty \theta e^{\theta T(t)} g(\theta) d\theta}{E_\Theta(e^{\Theta T(t)})} \\ &= \frac{E(\Theta \exp(\Theta T(t))}{E(\exp(\Theta T(t))}. \end{aligned} \tag{4.6}$$

By taking the derivative of the above expression with respect to t, it can be shown that

$$\frac{d}{dt} E(\Theta \mid T < t) = \tau(t) Var(\Theta \mid T < t) > 0. \tag{4.7}$$

Hence, $\frac{\tau^*(t)}{\tau(t)}$ is an increasing function of t. Also $\tau^*(t)$ and $\tau(t)$ can cross only at one point and the crossing point is a solution of the equation

$$\frac{d}{dt} M_\Theta[T(t)] = M_\Theta[T(t)] \tag{4.8}$$

Remark: If $E(\Theta) \geq 1$, then $\tau^*(t) \geq \tau(t)$, which is equivalent to $F^*(t)/F(t)$ is a non decreasing function of $t \geq 0$.

5. WEIBULL EXPONENTIATED FAMILY

5.1 Shape of the Failure Rate of Exponentiated Weibull Family

In order to study various shapes of the failure rate of our exponentiated Weibull model, we define the following four classes of failure rate for a general $r(t)$

1) If $\forall t > 0, r'(t) > 0$, then $F \in IFR$.
2) If $\forall t > 0, r'(t) < 0$, then $F \in DFR$.
3) If $\exists$ a $t^* > 0$, such that $r(t) < 0$ for all $t \in (0, t^*)$, $r'(t^*) = 0$ and $r'(t) > 0$ for all $t > t^*$. Then F is bathtub shaped failure rate i.e. $F \in B$.
4) If $\exists$ a $t^* > 0$, such that $r(t) > 0$ for all $t \in (0, t^*)$, $r'(t^*) = 0$ and $r'(t) < 0$ for all $t > t^*$. Then F is upside bathtub shaped failure rate i.e. $F \in U$.

For the determination of the above types of failure rates, see an excellent procedure given by Glaser (1980). Also for more general failure rates, see Gupta and Warren (2001).
Using the procedure described by Glaser, the following result is obtained, see Mudholkar (1995).

Theorem 5.1 *When the baseline distribution is Weibull,*

1) $F^* \in IFR$, *if* $\alpha \geq 1$ *and* $\alpha\theta \geq 1$
2) $F^* \in DFR$, *if* $\alpha \leq 1$ *and* $\alpha\theta \leq 1$
3) $F^* \in B$, *if* $\alpha > 1$ *and* $\alpha\theta < 1$
4) $F^* \in U$, *if* $\alpha < 1$ *and* $\alpha\theta > 1$.

The monotonicities are strict except for the negative exponential distribution corresponding to $\alpha = \theta = 1$.

5.2 Shape of the Probability Density Function

Let $z = e^{-(t/\sigma)^\alpha}$, we know $f^*(t) = \theta F^{\theta-1}(t) f(t)$. Then we have

$$f^{*'}(t) = \left[\frac{\alpha\theta F^{\theta-2}(t) f(t)}{t}\right][\psi_1(z) - \psi_2(z)], \tag{5.2.1}$$

where $\psi_1(z) = \left[\frac{1-z}{\theta} - (\theta-1)z\right] ln(z)$, $\psi_2(z) = \frac{1-z}{\theta}\left(\frac{1}{\alpha} - 1\right)$.

The shape of $f^*(t)$ can be studied by looking at the zeros of $f^{*'}(t)$. (5.2.1) shows that the first square bracket is always positive. Since z is a monotonic transformation of t, the zeros of $f^{*'}(t)$ correspond to the intersections between $\psi_1(z)$ and $\psi_2(z)$ over the interval $0 \le z \le 1$.

The $\psi_1(z)$ does not depend on α, and $\psi_1(0) = -\infty$, $\psi_1(1) = 0$;

$$\psi_1'(z) = \left[\frac{1-z}{\theta} - (\theta - 1)z\right]\frac{1}{z} + \left(1 - \theta - \frac{1}{\theta}\right)ln(z),\ \psi_1'(1) = 1 - \theta.$$

Thus, $\psi_1(z)$ increases with z when $\theta \le 1$, and is of the type U (unimodal) when $\theta > 1$.

The $\psi_2(z)$ is a straight line, and $\psi_2(0) = \frac{1}{\theta}\left(\frac{1}{\alpha} - 1\right)$, $\psi_2(1) = 0$, $\psi_2'(z) = \frac{1}{\theta}\left(1 - \frac{1}{\alpha}\right)$.

Thus, $\psi_2(z)$ is increasing when $\alpha > 1$ and decreasing when $\alpha < 1$.

So we have 6 cases about parametric characterization of pdf.

1. $\theta = 1$: The $F^*(t)$ reduces to a 2-parameter Weibull distribution; and its pdf idecreasing when $\alpha \le 1$, and is of the type U (unimodal) when $\alpha > 1$.

2. $\alpha = 1$: The $\psi_2(z)$ is a horizontal line. When $\theta \le 1$, $\psi_1(z) < 0, \psi_1'(1) > 0$, thus $\psi_1(z)$ and $\psi_2(z)$ don't intersect. When $\theta > 1$, $\psi_1'(1) < 0$, thus $\psi_1(z)$ and $\psi_2(z)$ intersect only once. So, the $f^*(t)$ is decreasing when $\theta \le 1$, and is of the type U (unimodal) when $\theta > 1$.

3. $\theta < 1, \alpha < 1$: $\psi_1(z) < 0, \psi_1'(1) > 0$ and $\psi_2(z) > 0$, they don't intersect. So $f^*(t)$ is decreasing.

4. $\theta < 1, \alpha > 1$: $\psi_1(z) < 0, \psi_1'(1) > 0$ and $\psi_2(z) < 0, \psi_2'(1) > 0$, as a result, $f^*(t)$ can be either decreasing (when $\psi_1'(1) \ge \psi_2'(1) \Leftrightarrow \alpha(1 + \theta^2 - \theta) \le 1$) or of the type U (unimodal) (when $\psi_1'(1) < \psi_2'(1) \Leftrightarrow \alpha(1 + \theta^2 - \theta) > 1$).

5. $\theta > 1, \alpha < 1$: $\psi_1'(1) < 0$, $\psi_1(z)$ is of the type U (unimodal), and $\psi_2(z) \ge 0, \psi_2'(z) < 0$.
   $f^*(t)$ can be either decreasing (when $\psi_1'(1) \ge \psi_2'(1) \Leftrightarrow \alpha(1 + \theta^2 - \theta) \le 1$) or of the type U (unimodal) (when $\psi_1'(1) < \psi_2'(1) \Leftrightarrow \alpha(1 + \theta^2 - \theta) > 1$).

6. $\theta > 1, \alpha > 1$: $\psi_1'(1) < 0$, $\psi_1(z)$ is of the type U (unimodal), and $\psi_2(z) \le 0, \psi_2'(z) > 0$, they intersect only once. $f^*(t)$ is always of the type U (unimodal).

From these 6 points, the conclusions are:

The pdf for the exponentiated Weibull family is:
- Decreasing when $\alpha(1+\theta^2-\theta)\le 1$ and $\alpha \ge 0, \theta \ge 0$.
- The type U (unimodal) when $\alpha(1+\theta^2-\theta)>1$ and $\alpha \ge 0, \theta \ge 0$.

Note that the expression for the functions $g_1(z)=(1-\theta\cdot z)ln(z)$ and $g_2(z)=\left(1-\frac{1}{\alpha}\right)\cdot(z-1)$ given in Jiang and Murthy (1999) are incorrect. Hence some of our conclusions are different from those of Jiang and Murthy (1999).

6. ANALYZING AN UNCENSORED DATA SET BY THE MODEL

In this section, we shall analyze Aarset (1987) data from reliability point of view and shall provide the MLE's and confidence bands for the failure rate and the survival function together with an estimate of the turning point of the failure rate.

Table 1. Lifetime of 50 Devices - Aarset Data

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18	18	18	18	18	21	32	36	40	45	46	47	50	55
60	63	63	67	67	67	67	72	75	79	82	82	83	84	84	84	85	85	85	85	85	86	86				

6.1 MLE of Parameters and Their Standard Errors

For exponentiated Weibull model, we have

$$f(t)=(\alpha/\sigma)(t/\sigma)^{\alpha-1}e^{-(t/\sigma)^\alpha},\ F(t)=1-e^{-(t/\sigma)^\alpha}.$$

$$f^*(t)=(\alpha\theta/\sigma)\left[1-e^{-(t/\sigma)^\alpha}\right]^{\theta-1}e^{-(t/\sigma)^\alpha}(t/\sigma)^{\alpha-1},\ F^*(t)=\left[1-e^{-(t/\sigma)^\alpha}\right]^\theta.$$

α,θ,σ are positive parameters; σ is a true scale parameter.

Given a random sample of size N from an exponentiated Weibull distribution, its parameters can be estimated by the ML method as follows.

Let

$$L=\prod_{i=1}^{N} f^*(t_i)$$

$$l=\ln L = N\ln(\alpha\theta/\sigma)+(\theta-1)\sum_{i=1}^{N}\ln(g(t_i))-\sum_{i=1}^{N}(t_i/\sigma)^\alpha+(\alpha-1)\sum_{i=1}^{N}\ln(t_i/\sigma), \quad (6.1.1)$$

where $g(t_i)=1-e^{-(t_i/\sigma)^\alpha}$.

The likelihood equations are given by

$$0 = \partial l/\partial \alpha = N/\alpha + (\theta - 1)\sum_{i=1}^{N} g_{\alpha}(t_i)/g(t_i) - \sum_{i=1}^{N}(t_i/\sigma)^{\alpha}\ln(t_i/\sigma) + \sum_{i=1}^{N}\ln(t_i/\sigma) \quad (6.1.2)$$

$$0 = \partial l/\partial \theta = N/\theta + \sum_{i=1}^{N}\ln(g(t_i)) \quad (6.1.3)$$

$$0 = \partial l/\partial \sigma = -(N\alpha/\sigma) + (\theta - 1)\sum_{i=1}^{N} g_{\sigma}(t_i)/g(t_i) + (\alpha/\sigma)\sum_{i=1}^{N}(t_i/\sigma)^{\alpha} \quad (6.1.4)$$

where $g_{\alpha}(t_i) = e^{-(t_i/\sigma)^{\alpha}}(t_i/\sigma)^{\alpha}\ln(t_i/\sigma)$, $g_{\sigma}(t_i) = -(\alpha/\sigma)e^{-(t_i/\sigma)^{\alpha}}(t_i/\sigma)^{\alpha}$.

The above three nonlinear equations are to be solved to obtain MLEs $\hat{\alpha},\hat{\theta},\hat{\sigma}$ of the parameters α,θ,σ.

The standard error (SE) of the estimators $\hat{\alpha},\hat{\theta},\hat{\sigma}$ can be obtained from the sample information matrix, see Appendix A.

The MLE's of the parameters are obtained as $\hat{\alpha} = 4.956$, $\hat{\theta} = 0.139$, $\hat{\sigma} = 91.164$.
The observed information matrix is given by

$$I_0 = \begin{pmatrix} 2.1367 & 71.0000 & 0.0492 \\ 71.0000 & 2587.8578 & 2.4333 \\ 0.0492 & 2.4333 & 0.0225 \end{pmatrix}$$

The variance-covariance matrix is given by

$$I_0^{-1} = \begin{pmatrix} 5.7636 & -0.1628 & 5.0073 \\ -0.1628 & 0.0050 & -0.1880 \\ 5.0073 & -0.1880 & 53.8257 \end{pmatrix}$$

So, the standard errors of the estimates $\hat{\alpha},\hat{\theta},\hat{\sigma}$ are $2.4007, 0.0707, 7.3366$ respectively.

6.2 Estimate and Confidence Bands of Survival Function

For our proposed model, the survival function

$$R(t) = P(X > t) = \bar{F}^{*}(t) = 1 - \left[1 - e^{-(t/\sigma)^{\alpha}}\right]^{\theta}.$$

We estimate $R(t)$ by $\hat{R}(t) = 1 - \left[1 - e^{-(t/\hat{\sigma})^{\hat{\alpha}}}\right]^{\hat{\theta}}$.

So, the 95% asymptotic confidence interval for $R(t)$ is

$$\hat{R}(t) \pm 1.96\sqrt{Var(\hat{R}(t))}$$

Now, we try to find $Var\hat{R}(t)$. Since $\hat{\alpha},\hat{\theta},\hat{\sigma}$ are estimates of α,θ,σ respectively, and $E(\hat{\alpha}) = \alpha$, $E(\hat{\theta}) = \theta$, $E(\hat{\sigma}) = \sigma$, we can use Taylor series to approximate the variance of $\hat{R}(t)$. See Casella and Berger. (1990, page329)

$$Var\left(\hat{R}(t)\right) \approx \sum_{i=1}^{3}\left(\hat{R}_i\right)^2 Var\left(X_i\right) + 2\sum_{i>j}\hat{R}_i\hat{R}_j Cov\left(X_i, X_j\right), \tag{6.2.1}$$

where $\hat{R}_1 = \frac{\partial}{\partial\alpha}R(t)|_{\hat{\alpha},\hat{\theta},\hat{\sigma}}$, $\hat{R}_2 = \frac{\partial}{\partial\theta}R(t)|_{\hat{\alpha},\hat{\theta},\hat{\sigma}}$, $\hat{R}_3 = \frac{\partial}{\partial\sigma}R(t)|_{\hat{\alpha},\hat{\theta},\hat{\sigma}}$, $X_1 = \hat{\alpha}$, $X_2 = \hat{\theta}$, $X_3 = \hat{\sigma}$.

$\hat{R}_1 = -\hat{\theta}\left(1 - e^{-(t/\hat{\sigma})^{\hat{\alpha}}}\right)^{\hat{\theta}-1} e^{-(t/\hat{\sigma})^{\hat{\alpha}}} \left(t/\hat{\sigma}\right)^{\hat{\alpha}} ln\left(t/\hat{\sigma}\right)$,

$\hat{R}_2 = -\left(1 - e^{-(t/\hat{\sigma})^{\hat{\alpha}}}\right)^{\hat{\theta}} ln\left(1 - e^{-(t/\hat{\sigma})^{\hat{\alpha}}}\right)$,

$\hat{R}_3 = \left(\hat{\alpha}\hat{\theta}/\hat{\sigma}\right)\left(1 - e^{-(t/\hat{\sigma})^{\hat{\alpha}}}\right)^{\hat{\theta}-1} e^{-(t/\hat{\sigma})^{\hat{\alpha}}} \left(t/\hat{\sigma}\right)^{\hat{\alpha}}$.

From the variance-covariance matrix, we know $Var\left(X_i\right)$ and $Cov\left(X_i, X_j\right)$, then we can get $Var\left(\hat{R}(t)\right)$ by (6.2.1). The confidence band for $R(t)$ for Aarset data is given in Figure 1.

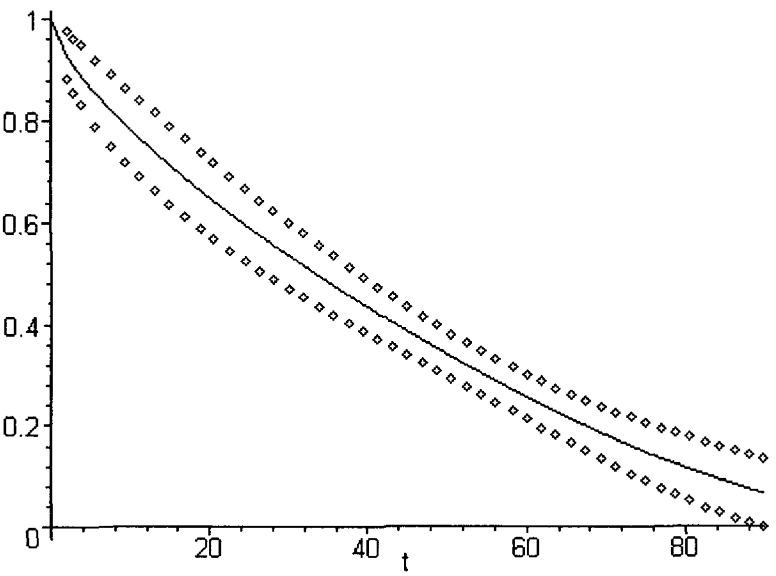

Figure 1. Confidence Bands of Survival Function

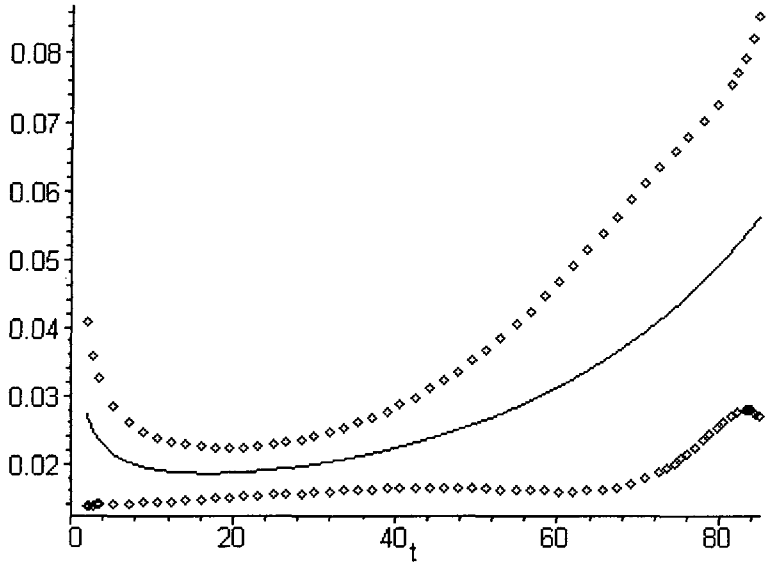

Figure 2. Confidence Bands of Failure Rate Function

6.3 Estimate and Confidence Bands of Failure Rate

For our exponentiated Weibull model, the failure rate function is given by

$$r(t)=\frac{f^*(t)}{1-F^*(t)}=(\alpha\theta/\sigma)\left(1-e^{-(t/\sigma)^{\alpha}}\right)^{\theta-1}e^{-(t/\sigma)^{\alpha}}(t/\sigma)^{\alpha-1}\Big/\left(1-\left(1-e^{-(t/\sigma)^{\alpha}}\right)^{\theta}\right)$$

$$=(-\theta\sigma/t)g^{\theta-1}(t)g_{\sigma}(t)/\left(1-g^{\theta}(t)\right)$$

We estimate $r(t)$ by $\hat{r}(t)=\left(\hat{\alpha}\hat{\theta}/\hat{\sigma}\right)\left(1-e^{-(t/\hat{\sigma})^{\hat{\alpha}}}\right)^{\hat{\theta}-1}e^{-(t/\hat{\sigma})^{\hat{\alpha}}}(t/\hat{\sigma})^{\hat{\alpha}-1}\Big/\left(1-\left(1-e^{-(t/\hat{\sigma})^{\hat{\alpha}}}\right)^{\hat{\theta}}\right)$

$$=\left(-\hat{\theta}\hat{\sigma}/t\right)g^{\hat{\theta}-1}(t)g_{\hat{\sigma}}(t)/\left(1-g^{\hat{\theta}}(t)\right)$$

So, the 95% asymptotic confidence interval for $r(t)$ is

$$\hat{r}(t)\pm 1.96\sqrt{Var(\hat{r}(t))}$$

Now, we try to find $Var(\hat{r}(t))$. Since $\hat{\alpha},\hat{\theta},\hat{\sigma}$ are the MLE's of α,θ,σ respectively, and $E(\hat{\alpha})=\alpha$, $E(\hat{\theta})=\theta$, $E(\hat{\sigma})=\sigma$, we can use Taylor series to approximate the variance of $\hat{r}(t)$. See Casella and Berger. (1990 page329)

$$Var(\hat{r}(t))\approx\sum_{i=1}^{3}[\hat{r}_i]^2 Var(X_i)+2\sum_{i>j}\hat{r}_i\hat{r}_j Cov(X_i,X_j), \qquad (6.3.1)$$

where $\hat{r}_1=\frac{\partial}{\partial\alpha}r(t)|_{\hat{\alpha},\hat{\theta},\hat{\sigma}}$, $\hat{r}_2=\frac{\partial}{\partial\theta}r(t)|_{\hat{\alpha},\hat{\theta},\hat{\sigma}}$, $\hat{r}_3=\frac{\partial}{\partial\sigma}r(t)|_{\hat{\alpha},\hat{\theta},\hat{\sigma}}$, $X_1=\hat{\alpha}$, $X_2=\hat{\theta}$, $X_3=\hat{\sigma}$.

$$\hat{r}_1 = \left(g^{\hat{\theta}-2}(t)g_{\hat{\alpha}}(t)g_{\hat{\sigma}}(t)\left(\hat{\theta}-1+g^{\hat{\theta}}(t)\right)+\left(g^{\hat{\theta}-1}(t)/\hat{\alpha}\hat{\sigma}\right)\left(1-g^{\hat{\theta}}(t)\right)\left(g_{\hat{\alpha}}(t)\left(\hat{\sigma}-\hat{\alpha}^2-\hat{\alpha}^2(t/\hat{\sigma})^{\hat{\alpha}}\right)\right)\right)$$
$$\left(-\hat{\theta}\hat{\sigma}/t\right)/\left(1-g^{\hat{\theta}}(t)\right)^2,$$
$$\hat{r}_2 = \left(-\hat{\sigma}g_{\hat{\sigma}}(t)/t\right)g^{\hat{\theta}-1}(t)\left(1-g^{\hat{\theta}}(t)+\hat{\theta}\,ln(g(t))\right)/\left(1-g^{\hat{\theta}}(t)\right)^2,$$
$$\hat{r}_3 = -\left(\hat{\theta}/t\right)\left(\hat{\sigma}g^{\hat{\theta}-2}(t)g_{\hat{\sigma}}^2(t)\left(\hat{\theta}-1+g^{\hat{\theta}}(t)\right)+\hat{\alpha}g^{\hat{\theta}-1}(t)g_{\hat{\sigma}}(t)\left((t/\hat{\sigma})^{\hat{\alpha}}-1\right)\left(1-g^{\hat{\theta}}(t)\right)\right)/\left(1-g^{\hat{\theta}}(t)\right)^2$$

From the variance-covariance matrix, we know $Var(X_i)$ and $Cov(X_i, X_j)$, then we can get $Var(\hat{r}(t))$ by (6.3.1). The confidence band for $\hat{r}(t)$ for our data set is given in Figure 2.

6.4 Estimate of Turning Point of Failure Rate Function

To find the estimate of turning point, we need to solve $\hat{r}'(t) = 0$. Since the expression for $\hat{r}'(t)$ is quite complex, we proceed as follows:

Define $\eta(t) = -f'(t)/f(t)$.

Then it can be verified that $\hat{r}'(t)/\hat{r}(t) = f'(t)/\hat{f}(t) + \hat{f}(t)/\hat{R}(t) = r(t) - \eta'(t)$. Thus we solve the equation

$$\hat{r}'(t) = \hat{\eta}(t) \text{ or } \hat{f}^2(t) + \hat{f}'(t)\left(1-\hat{F}(t)\right) = 0. \tag{6.4.1}$$

Solving (6.4.1) numerically, we get the estimate of turning point $\hat{t}_0 = 16.7761$. The graph of the failure rate is given in Figure 3. It is of the type B.

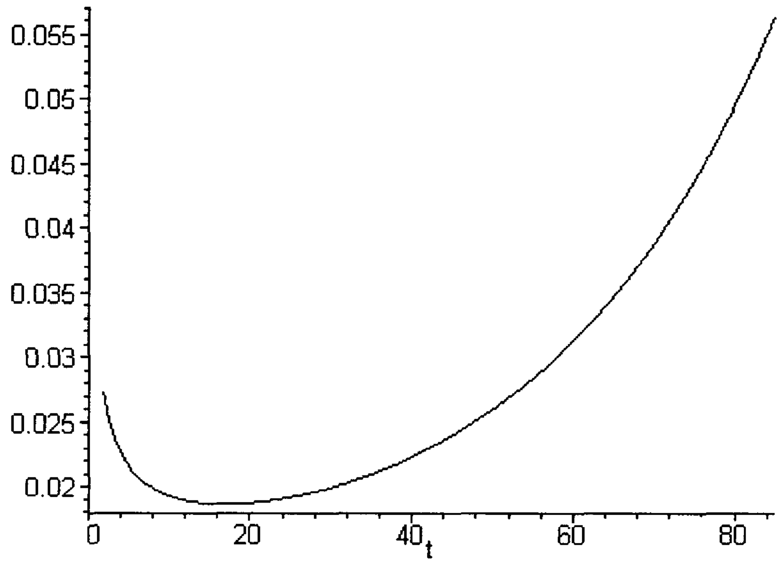

Figure 3. Graph of Failure Rate Function

6.5 Goodness of Fit

6.5.1 Likelihood Ratio Test

Let T_i, $i = 1,...N$ be a random sample from a distribution with pdf $f(t_i,\Theta)$, where $\Theta = (\theta_1,...\theta_k)'$ is a vector of unknown parameters. The likelihood function for Θ is

$$L = \prod_{i=1}^{N} f(t_i,\Theta).$$

Under null hypothesis H_0 : $\Theta = \Theta_0$

$$\Lambda = -2\,log\left(\frac{L(\Theta_0)}{L(\hat{\Theta})}\right) \tag{6.5.1a}$$

is the test statistic and is asymptotically distributed as $\chi^2(k)$, where $\hat{\Theta}$ is maximum likelihood estimate of Θ .
Tests and estimates for subset of the Θ_i 's can also be obtained: Suppose that Θ is partitioned as $\Theta = (\Theta_1,\Theta_2)'$, where Θ_1 is $p\times 1$ and Θ_2 is $(k-p)\times 1$. We consider $H_0 : \Theta_1 = \Theta_{10}$, then

$$\Lambda = -2\,log\left(\frac{L[\Theta_{10},\tilde{\Theta}_2(\Theta_{10})]}{L(\hat{\Theta}_1,\hat{\Theta}_2)}\right) \tag{6.5.1b}$$

is asympototically distributed as $\chi^2(p)$.

In our case, $\Theta = (\alpha,\theta,\sigma)'$. We want to test the goodness of fit of the proposed model under null hypothesis $H_0 : \theta = 1$. Using IMSL routine DNEQNF, by writing a FORTRAN program, solve a system of two non-linear equations about α and σ (because $\theta = 1$), we get restricted likelihood estimates $\tilde{\alpha} = 0.949, \tilde{\sigma} = 44.913$. We get $\Lambda = 24.3686 > 3.84$ which is $\chi^2_{0.05}(1)$, so we reject null hypothesis, which is Weibull distribution.

6.5.2 Wald Test

Let T_i, $i = 1,...N$ be a random sample from a distribution with pdf $f(t_i,\Theta)$, where $\Theta = (\theta_1,...\theta_k)'$ is a vector of unknown parameters and $\hat{\Theta} = (\hat{\theta}_1,...\hat{\theta}_k)'$ is the unrestricted M.L.E. of $\Theta = (\theta_1,...\theta_k)'$.
Suppose that the null hypothesis $H_0 : h(\Theta) = [h_1(\Theta),...h_r(\Theta)]' = 0$.

We define H as a $r \times k$ matrix with entries $\frac{\partial h_j(\Theta)}{\partial \theta_i}, i = 1,2,...k; j = 1,2,...r$ and $I(\Theta)$ as the Fisher information matrix with entries $I_{ij}(\Theta) = E\left(-\partial^2 \ln L/\partial\theta_i\partial\theta_j\right)$, $i, j = 1,2,...k$.

The test statistic, $W = h'(\hat{\Theta})\left[\hat{H}' I^{-1}(\hat{\Theta})\hat{H}\right]^{-1} h(\hat{\Theta})$, is given by Wald, see Silvey (1975).

Furthermore, W is asymptotically distributed as $\chi^2(r)$.

In our case, $\Theta = (\alpha, \theta, \sigma)'$. We want to test the goodness of fit of the proposed model under null hypothesis $H_0 : \theta = 1$. Obviously, our $r = 1$, $h_1(\Theta) = \theta - 1 = 0$. Then we know $\hat{H}' = [0,1,0]$, $h'(\hat{\Theta}) = \hat{\theta} - 1$. Using the estimates obtained in section 6.1, we get $W = 148.2642 > 3.84 = \chi^2_{0.05}(1)$, so we reject null hypothesis, and conclude that Weibull distribution does not fit the data well.

7. ANALYZING A CENSORED DATA SET BY THE MODEL

The flexibility of the proposed exponentiated Weibull family is further emphasized by using it to model a censored data set. Given a random sample of size N from an exponentiated Weibull distribution. Let $T_1, T_2, ...T_K$ be uncensored, and $T_{K+1}, T_{K+2}, ...T_N$ be censored observations. We will analyze the following data given by Efron (1988).

Tabl 2. Survival Times(in days) for the Patients in Arm A of the Head-and-Neck-Cancer Trial Note: From Efron (1988)

7, 34, 42, 63, 64, 74+, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 173, 176, 185+, 218, 225, 241, 248, 273, 277, 279+ 297, 319+, 405, 417, 420, 440, 523, 523+, 583, 594, 1101, 1116+, 1146, 1226+, 1349+, 1412+, 1417

+ indicates observations lost to follow up

7.1 MLE of Parameters and Their Standard Errors

For exponentiated Weibull model, we have

$$f(t) = (\alpha/\sigma)(t/\sigma)^{\alpha-1} e^{-(t/\sigma)^\alpha}, \quad F(t) = 1 - e^{-(t/\sigma)^\alpha}.$$

$$f^*(t) = (\alpha\theta/\sigma)\left[1 - e^{-(t/\sigma)^\alpha}\right]^{\theta-1} e^{-(t/\sigma)^\alpha} (t/\sigma)^{\alpha-1}, \quad F^*(t) = \left[1 - e^{-(t/\sigma)^\alpha}\right]^\theta.$$

α, θ, σ are positive parameters; σ is a true scale parameter.

Then the likelihood function is given by

$$L = \prod_{i=1}^{K} f^*(t_i) \prod_{i=K+1}^{N} (1 - F^*(t_i)) = \prod_{i=1}^{K} (\alpha\theta/\sigma) g^{\theta-1}(t_i) e^{-(t_i/\sigma)^\alpha} (t_i/\sigma)^{\alpha-1} \prod_{i=K+1}^{N} (1 - g^\theta(t_i)),$$

where $g(t_i)=1-e^{-(t_i/\sigma)^\alpha}$.

Log likelihood function is given by

$$l=\ln L=K\ln(\alpha\theta/\sigma)+(\theta-1)\sum_{i=1}^{K}\ln(g(t_i))-\sum_{i=1}^{K}(t_i/\sigma)^\alpha$$
$$+(\alpha-1)\sum_{i=1}^{K}ln(t_i/\sigma)+\sum_{i=k+1}^{N}ln(1-g^\theta(t_i)) \tag{7.1.1}$$

Then the likelihood equations are given by

$$0=\partial l/\partial\alpha=K/\alpha+(\theta-1)\sum_{i=1}^{K}g_\alpha(t_i)/g(t_i)-\sum_{i=1}^{K}(t_i/\sigma)^\alpha ln(t_i/\sigma)+\sum_{i=1}^{K}ln(t_i/\sigma)$$
$$-\sum_{i=K+1}^{N}\theta g^{\theta-1}(t_i)g_\alpha(t_i)/(1-g^\theta(t_i)), \tag{7.1.2}$$

$$0=\partial l/\partial\theta=K/\theta+\sum_{i=1}^{K}ln(g(t_i))-\sum_{i=K+1}^{N}g^\theta(t_i)ln(g(t_i))/(1-g^\theta(t_i)), \tag{7.1.3}$$

$$0=\partial l/\partial\sigma=-(K\alpha/\sigma)+(\theta-1)\sum_{i=1}^{K}g_\sigma(t_i)/g(t_i)+(\alpha/\sigma)\sum_{i=1}^{K}(t_i/\sigma)^\alpha$$
$$-\sum_{i=K+1}^{N}\theta g^{\theta-1}(t_i)g_\sigma(t_i)/(1-g^\theta(t_i)), \tag{7.1.4}$$

where $g_\alpha(t_i)=e^{-(t_i/\sigma)^\alpha}(t_i/\sigma)^\alpha\ln(t_i/\sigma)$, $g_\sigma(t_i)=-(\alpha/\sigma)e^{-(t_i/\sigma)^\alpha}(t_i/\sigma)^\alpha$.

Then the MLE's for the exponentiated Weibull can be obtained by solving the above three ML equations for α,θ,σ.

The standard error (SE) of the estimators $\hat{\alpha},\hat{\theta},\hat{\sigma}$ based on the sample can be obtained from the variance-covariance matrix, see Appendix B.

Using the method described in section 6, we get MLEs:

$$\hat{\alpha}=0.3105,\ \hat{\theta}=15.1850,\ \hat{\sigma}=0.2117.$$

The observed information matrix is given by

$$I_0=\begin{pmatrix}25873.1778 & -1.8462 & -7042.0836\\ -1.8462 & 0.2213 & 0.5553\\ -7042.0836 & 0.5553 & 3910.1092\end{pmatrix}$$

The variance-covariance matrix is given by

$$I_0^{-1} = \begin{pmatrix} 0.000076 & 0.00029 & 0.00014 \\ 0.00029 & 4.52147 & -0.00012 \\ 0.00014 & -0.00012 & 0.00050 \end{pmatrix}$$

So, the standard errors of the estimates $\hat{\alpha}, \hat{\theta}, \hat{\sigma}$ are $0.0087, 2.1264, 0.0224$ respectively.

7.2 Estimate and Confidence Bands of Survival Function

For our model, the survival function is $R(t) = P(X > t) = \overline{F}^*(t) = 1 - \left(1 - e^{-(t/\sigma)^\alpha}\right)^\theta$. We estimate $R(t)$ by $\hat{R}(t) = 1 - \left(1 - e^{-(t/\hat{\sigma})^{\hat{\alpha}}}\right)^{\hat{\theta}}$. So, the 95% asymptotic confidence interval for $R(t)$ is $\hat{R}(t) \pm 1.96\sqrt{Var\left(\hat{R}(t)\right)}$.

Proceeding as in section 6, we can obtain a confidence interval for $\hat{R}(t)$. The confidence band for $\hat{R}(t)$ is given in Figure 4.

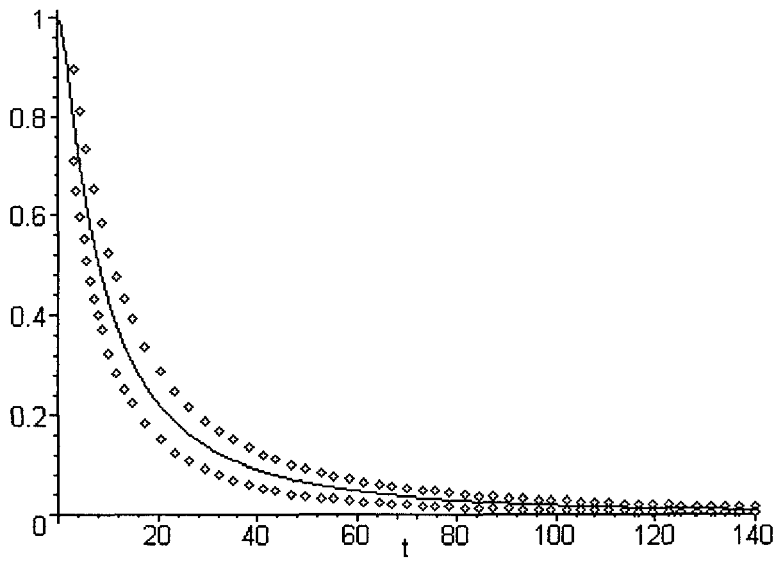

Figure 4. Confidence Bands of Survival Function

7.3 Estimate and Confidence Bands of Failure Rate

Proceeding as before, we can obtain an asymptotic confidence interval for $\hat{r}(t)$. The confidence bands for $\hat{r}(t)$ is given in Figure 5.

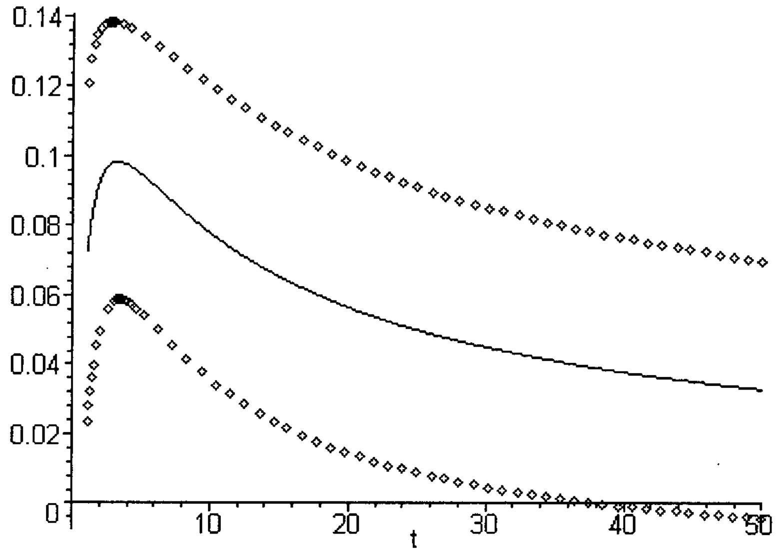

Figure 5. Confidence Bands of Failure Rate Function

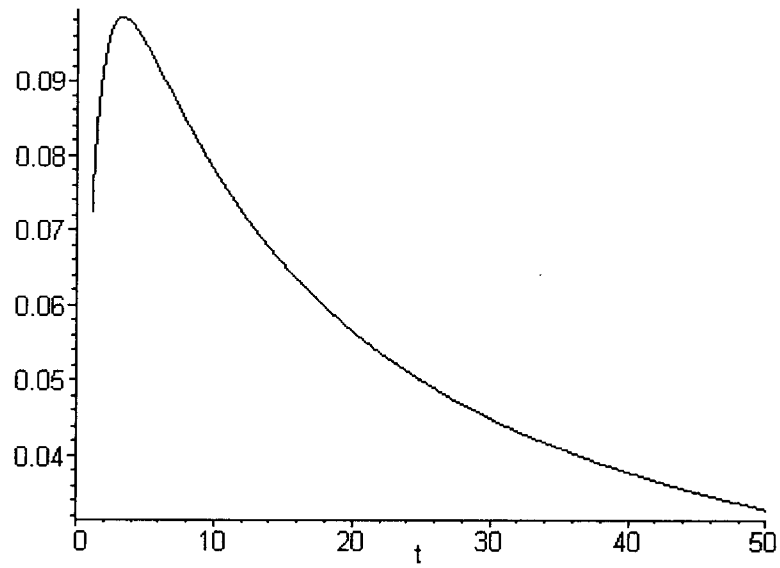

Figure 6. Graph of Failure Rate Function

7.4 Estimate of Turning Point of Failure Rate Function

Proceeding as before, we solve $\hat{r}(t) = \hat{\eta}(t)$. The estimate of turning point is given by $\hat{t}_0 = 3.30156$. The estimate of the failure rate is given in Figure 6. It is of the type U.

7.5 Goodness of Fit

In this case, the restricted maximum likelihood estimates are $\tilde{\alpha} = 0.930, \tilde{\sigma} = 14.024$.

1) The value of the likelihood ratio test statistic $\Lambda = 7.7052 > 3.84$ which is $\chi^2_{0.05}(1)$, so we reject null hypothesis, which is Weibull distribution.

2) The value of the Wald statistic is given by $W = 44.5019 > 3.84 = \chi^2_{0.05}(1)$, so we reject null hypothesis, and conclude that Weibull distribution does not fit the data well.

8. CONCLUSIONS AND RECOMMENDATIONS

The discussion in this paper shows that in practice many times the data should be modeled by a distribution function which is the power of a baseline distribution function.

1) In the case of Aarset data (uncersored) $\hat{\theta} = 0.139$.
2) In the case of Efron data (censored) $\hat{\theta} = 15.185$.

We can see that the proposed model is flexible enough to accommodate monotonic as well as non-monotonic failure rates even though the baseline failure rate is monotonic. In practice, modeling survival data by non-monotonic failure rate is desirable. For example, when the course of the disease is such that mortality reaches a peak after some finite period and then slowly declines.

APPENDIX A

From equations (6.1.2), (6.1.3), (6.1.4), we get the following 9 second-order partial derivatives,

$$\partial^2 l/\partial\alpha^2 = -N/\alpha^2 + (\theta-1)\sum_{i=1}^{N} g_\alpha(t_i)\left(g(t_i)\ln(t_i/\sigma)\left(1-(t_i/\sigma)^\alpha\right) - g_\alpha(t_i)\right)/g^2(t_i)$$

$$-\sum_{i=1}^{N}\ln^2(t_i/\sigma)(t_i/\sigma)^\alpha$$

$$\partial^2 l/\partial\alpha\partial\theta = \sum_{i=1}^{N} g_\alpha(t_i)/g(t_i)$$

$$\partial^2 l/\partial\alpha\partial\sigma = \sum_{i=1}^{N}\left((\alpha/\sigma)(t_i/\sigma)^\alpha \ln(t_i/\sigma) + (1/\sigma)(t_i/\sigma)^\alpha\right) - \sum_{i=1}^{N} 1/\sigma + (\theta-1)$$

$$\sum_{i=1}^{N}\left(\alpha^2(t_i/\sigma)^\alpha g_\alpha(t_i)g(t_i) - \alpha^2 g_\alpha(t_i)g(t_i) + \sigma g_\sigma(t_i)g(t_i) - \alpha\sigma g_\alpha(t_i)g_\sigma(t_i)\right)/\alpha\sigma g^2(t_i)$$

$$\partial^2 l/\partial\theta\partial\alpha = \sum_{i=1}^{N} g_\alpha(t_i)/g(t_i)$$

$$\partial^2 l/\partial\theta^2 = -N/\theta^2$$

$$\partial^2 l/\partial\theta\partial\sigma = \sum_{i=1}^{N} g_\sigma(t_i)/g(t_i)$$

$$\partial^2 l/\partial\sigma\partial\alpha = -N/\sigma + (1/\sigma)\sum_{i=1}^{N}(t_i/\sigma)^\alpha + (\alpha/\sigma)\sum_{i=1}^{N}(t_i/\sigma)^\alpha \ln(t_i/\sigma) + (\theta-1)$$
$$\sum_{i=1}^{N}\{(\alpha/\sigma)((t_i/\sigma)^\alpha - 1)g_\alpha(t_i)g(t_i) + (1/\alpha)g_\sigma(t_i)g(t_i) - g_\alpha(t_i)g_\sigma(t_i)\}/g^2(t_i)$$

$$\partial^2 l/\partial\sigma\partial\theta = \sum_{i=1}^{N} g_\sigma(t_i)/g(t_i)$$

$$\partial^2 l/\partial\sigma^2 = N\alpha/\sigma^2 - (\alpha/\sigma^2 + \alpha^2/\sigma^2)\sum_{i=1}^{N}(t_i/\sigma)^\alpha + (\theta-1)$$
$$\sum_{i=1}^{N}(g(t_i)g_\sigma(t_i)((\alpha/\sigma)(t_i/\sigma)^\alpha - 1/\sigma - \alpha/\sigma) - g_\sigma^{\,2}(t_i))/g^2(t_i)$$

Then we can get the information matrix as

$$I = -E\begin{pmatrix} \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\theta} & \frac{\partial^2 l}{\partial\alpha\partial\sigma} \\ \frac{\partial^2 l}{\partial\theta\partial\alpha} & \frac{\partial^2 l}{\partial\theta^2} & \frac{\partial^2 l}{\partial\theta\partial\sigma} \\ \frac{\partial^2 l}{\partial\sigma\partial\alpha} & \frac{\partial^2 l}{\partial\sigma\partial\theta} & \frac{\partial^2 l}{\partial\sigma^2} \end{pmatrix}$$

The matrix I is called the Fisher (or expected) information matrix. Obviously, it is a symmetric matrix. Obtaining the actual Fisher information matrix is not easy in most practical problems including the problem at hand. In such cases we obtain the observed information matrix (defined by I_0) instead,

$$I_0 = -\begin{pmatrix} \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\theta} & \frac{\partial^2 l}{\partial\alpha\partial\sigma} \\ \frac{\partial^2 l}{\partial\theta\partial\alpha} & \frac{\partial^2 l}{\partial\theta^2} & \frac{\partial^2 l}{\partial\theta\partial\sigma} \\ \frac{\partial^2 l}{\partial\sigma\partial\alpha} & \frac{\partial^2 l}{\partial\sigma\partial\theta} & \frac{\partial^2 l}{\partial\sigma^2} \end{pmatrix}_{|\alpha,\theta,\sigma=\hat{\alpha},\hat{\theta},\hat{\sigma}}$$

The inverse matrix of I_0 provides the standard errors of the estimates.

APPENDIX B

For the censored data, the second order partial derivatives are given by

$$\partial^2 l/\partial\alpha^2 = -K/\alpha^2 + (\theta-1)\sum_{i=1}^{K} g_\alpha(t_i)\left(g(t_i)\ln(t_i/\sigma)\left(1-(t_i/\sigma)^\alpha\right) - g_\alpha(t_i)\right)/g^2(t_i)$$

$$-\sum_{i=1}^{K}\ln^2(t_i/\sigma)(t_i/\sigma)^\alpha - \theta^2\sum_{i=K+1}^{N} g_\alpha^2(t_i)g^{\theta-2}(t_i)/\left(1-g^\theta(t_i)\right)^2$$

$$-\theta\sum_{i=K+1}^{N}\left(g_\alpha(t_i)g^{\theta-1}(t_i)\ln(t_i/\sigma)\left(1-(t_i/\sigma)^\alpha\right) - g^{\theta-2}(t_i)g_\alpha^2(t_i)\right)/\left(1-g^\theta(t_i)\right)$$

$$\partial^2 l/\partial\alpha\partial\theta = \sum_{i=1}^{K} g_\alpha(t_i)/g(t_i) - \sum_{i=K+1}^{N} g_\alpha(t_i)g^{\theta-1}(t_i)/\left(1-g^\theta(t_i)\right)$$

$$-\theta\sum_{i=K+1}^{N} g_\alpha(t_i)g^{\theta-1}(t_i)\ln(g(t_i))/\left(1-g^\theta(t_i)\right)^2$$

$$\partial^2 l/\partial\alpha\partial\sigma = \sum_{i=1}^{K}(1/\sigma)(t_i/\sigma)^\alpha\left(1+\alpha\ln(t_i/\sigma)\right) - K/\sigma$$

$$+(\theta-1)\sum_{i=1}^{K}\left(g(t_i)\left(\alpha g_\alpha(t_i)\left((t_i/\sigma)^\alpha - 1\right)/\sigma + g_\sigma(t_i)/\alpha\right) - g_\alpha(t_i)g_\sigma(t_i)\right)/g^2(t_i) - (\theta/\alpha\sigma)$$

$$\sum_{i=K+1}^{N} g^{\theta-2}(t_i)\left(\alpha^2 g_\alpha(t_i)g(t_i)\left((t_i/\sigma)^\alpha - 1\right) + \sigma g_\sigma(t_i)g(t_i) - \alpha\sigma g_\sigma(t_i)g_\alpha(t_i)\right)/\left(1-g^\theta(t_i)\right)$$

$$-\theta^2\sum_{i=K+1}^{N} g^{\theta-2}(t_i)g_\sigma(t_i)g_\alpha(t_i)/\left(1-g^\theta(t_i)\right)^2$$

$$\partial^2 l/\partial\theta\partial\alpha = \sum_{i=1}^{K} g_\alpha(t_i)/g(t_i) - \sum_{i=K+1}^{N} g^{\theta-1}(t_i)g_\alpha(t_i)\left(1-g^\theta(t_i)+\theta\ln(g(t_i))\right)/\left(1-g^\theta(t_i)\right)^2$$

$$\partial^2 l/\partial\theta^2 = -K/\theta^2 - \sum_{i=K+1}^{N} g^\theta(t_i)\ln^2(g(t_i))/\left(1-g^\theta(t_i)\right)^2$$

$$\partial^2 l/\partial\theta\partial\sigma = \sum_{i=1}^{K} g_\sigma(t_i)/g(t_i)$$

$$-\sum_{i=K+1}^{N} g^{\theta-1}(t_i)g_\sigma(t_i)\left(\theta\ln(g(t_i)) + 1 - g^\theta(t_i)\right)/\left(1-g^\theta(t_i)\right)^2$$

$$\partial^2 l/\partial\sigma\partial\alpha = -K/\sigma + (1/\sigma)\sum_{i=1}^{K}(t_i/\sigma)^{\alpha} + (\alpha/\sigma)\sum_{i=1}^{K}(t_i/\sigma)^{\alpha}\ln(t_i/\sigma) + (\theta-1)$$

$$\sum_{i=1}^{K}\left((\alpha/\sigma)\left((t_i/\sigma)^{\alpha}-1\right)g_{\alpha}(t_i)g(t_i) + (1/\alpha)g_{\sigma}(t_i)g(t_i) - g_{\alpha}(t_i)g_{\sigma}(t_i)\right)/g^2(t_i) - \theta$$

$$\sum_{i=K+1}^{N}\left(g^{\theta-1}(t_i)\left((\alpha/\sigma)(t_i/\sigma)^{\alpha}g_{\alpha}(t_i) + g_{\sigma}(t_i)/\alpha - \alpha g_{\alpha}(t_i)/\sigma\right) - g^{\theta-2}(t_i)g_{\alpha}(t_i)g_{\sigma}(t_i)\right)/\left(1-g^{\theta}(t_i)\right)$$

$$-\theta^2\sum_{i=K+1}^{N}g^{\theta-2}(t_i)g_{\alpha}(t_i)g_{\sigma}(t_i)/\left(1-g^{\theta}(t_i)\right)^2$$

$$\partial^2 l/\partial\sigma\partial\theta = \sum_{i=1}^{K}g_{\sigma}(t_i)/g(t_i)$$

$$-\sum_{i=K+1}^{N}g^{\theta-1}(t_i)g_{\sigma}(t_i)\left(\theta\ln(g(t_i)) + 1 - g^{\theta}(t_i)\right)/\left(1-g^{\theta}(t_i)\right)^2$$

$$\partial^2 l/\partial\sigma^2 = K\alpha/\sigma^2 - \left(\alpha/\sigma^2 + \alpha^2/\sigma^2\right)\sum_{i=1}^{K}(t_i/\sigma)^{\alpha} + (\theta-1)$$

$$\sum_{i=1}^{K}\left(g(t_i)g_{\sigma}(t_i)\left((\alpha/\sigma)(t_i/\sigma)^{\alpha} - 1/\sigma - \alpha/\sigma\right) - g_{\sigma}^{2}(t_i)\right)/g^2(t_i)$$

$$-\theta\sum_{i=K+1}^{N}\left(g^{\theta-1}(t_i)g_{\sigma}(t_i)\left(\alpha(t_i/\sigma)^{\alpha}/\sigma - 1/\sigma - \alpha/\sigma\right) - g^{\theta-2}(t_i)g_{\sigma}^{2}(t_i)\right)/\left(1-g^{\theta}(t_i)\right)$$

$$-\theta^2\sum_{i=K+1}^{N}g^{\theta-2}(t_i)g_{\sigma}^{2}(t_i)/\left(1-g^{\theta}(t_i)\right)^2$$

From these the observed variance-covariance matrix can be obtained.

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