◆ **Research Paper**

Deriving a Probabilistic Model for Fatigue Life Based on Physical Failure Mechanism

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Abstract

A probabilistic model for fatigue life of a structural component is derived when the component is in a variable-amplitude loading environment. The physical mechanism which governs fatigue failure is used to model the fatigue life. Especially, the judgement of rotational symmetry in the-stress-intensity-factors results in the probability distribution for fatigue life. The probability distribution is related to the familiar truncated Gaussian distribution, which has a single parameter with a direct physical meaning.

1. Introduction

Many engineering structures such as bridges, airplanes and ships undergo dynamic loading. When a structural component is subjected to a variable-amplitude load below yielding stress, fatigue is the commonly identified cause of the component's failure.

Fatigue fracture is due to a process consisting of three successive phases [6, 7]: (1) crack initiation, (2) propagation and (3) final fracture. Cracks always exist; e.g., rough surfaces contain initial cracks. When a load is applied to a dominant initial crack, the crack propagates up to a critical length and final fracture begins. The crack propagation phase occupies a major portion of the component's life. The final fracture occurs when the unfractured cross-section is unable to sustain the maximum tensile stress of loading. This moment defines fatigue failure. However, the period of the propagation phase varies even for similar materials under similar loading conditions. Therefore, the moment of fatigue failure, or fatigue life, is uncertain.

To predict the variability of a component's fatigue life, probability distributions are used. Common examples used in practice include such two-parameter families of distributions as the lognormal, Weibull and Gamma. A physical motivation for these models is often lacking. This problem is compounded by the fact that, in practice, only small samples of data are available. This makes it difficult to identify the appropriate distribution using goodness-of-fit tests. This is especially true when the prediction of the "safe life" is

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of concern [4].

This paper proposes a new procedure to determine fatigue life distributions. It starts with the physical laws underlying the fatigue process. It is assumed that crack propagation under a variable-amplitude load shows relatively smooth and continuous behavior. This type of assumption was first put forward by Barsom [2]. It makes expressions for the crack-propagation behavior under a constant-amplitude load tractable for fatigue life modeling.

In Section 3, the physical relationship between propagation rate and stress range is used to express the distribution for fatigue life in terms of the distribution for stress range. This is useful in cases where a distribution for stress range is available. For instance, the Rayleigh distribution is common for stress range and the methods of Section 3 show which fatigue life distribution is consistent with it.

In Section 4, certain physical symmetries implied by the stress-intensity-factor range is used to derive a Bayesian statistical-parametric class for fatigue life. There, the truncated Gaussian model emerges with the root mean squared stress-intensity-factor range serving the role of the scale parameter.

Notation

c	crack length
Δc	crack increment or crack measurement unit
c_0 , c_f	initial crack length and critical crack length
N	number of cycles
A , m	empirical constants
σ	stress
$\Delta\sigma$	stress range
K	stress-intensity-factor
ΔK	stress-intensity-factor range
M^*	number of cycle blocks until fatigue failure
L	fatigue life in the number of cycles until failure

2. Crack Propagation

When a structural component is subjected to a constant-amplitude load, the following differential equation is commonly employed to describe crack propagation [3, 5, 10]:

$$\frac{dc}{dN} = A(\Delta K)^m. \qquad (1)$$

Here A is a constant which depends on the material and stress ratio, and m is an empirical constant depending on the material properties. This equation makes the crack-propagation rate dc/dN a function of the stress-intensity-factor range ΔK

A different function is found by using the relation $\Delta K = \alpha\, \Delta\sigma\sqrt{\pi c}$ employed in the theory of fracture mechanics [8]. Here α is the crack's geometrical factor. Substituting this in Equation 1, we find

$$\frac{dc}{dN} = A(\alpha\, \Delta\sigma\sqrt{\pi c})^m, \tag{2}$$

or that the crack-propagation rate is a function of stress range and crack length.

Assuming that the propagation behavior under a variable-amplitude load is smooth and continuous, either of the above two differential equations can be used to model fatigue life. Each leads to a distinct life model. These will be presented in the next two sections.

In the literature [5], it is the change in crack length per cycle that is considered random. Specifically, cycle blocks consisting of several loading cycles are assumed and the change in crack length during each cycle block is random. The problem is that there is a strong and mathematically complicated dependence between these random variables (see Equation 2). To avoid this problem, the number of cycles requiring a unit crack increment is instead used as the random variable. This variable is equally effective and, at the same time, relates in a mathematically simple fashion to the pertinent quantities of the fatigue process.

3. Life Model Based on Stress Range

When we first consider the description of the fatigue-propagation behavior offered by Equation 2, fatigue life L becomes the number of cycles necessary to cause fatigue failure. To determine this, the propagation of the crack can be discretized into unit crack increments. These increments could be a convenient measuring unit. Let ΔN_i be the number of cycles containing in the ith unit increment. Each ΔN_i is contained in a cycle block. The number of cycle blocks M until fatigue failure is determined by the initial and critical crack length as follows: $M = \frac{c_f - c_0}{\Delta c}$.Let M^* be the greatest integer less than or equal to the real number M. M^* reflects a conservative prediction of fatigue life. Then, the fatigue life random variable L can be written as the sum of the number of cycles within each cycle block:

$$L = \Delta N_1 + \Delta N_2 + \cdots + \Delta N_{M^*} \quad . \tag{3}$$

We now proceed to express the fatigue life L in terms of the stress range $\Delta\sigma$ The difference equation corresponding to Equation 2 is

$$\frac{\Delta c}{\Delta N} = A(\alpha\, \Delta\sigma\sqrt{\pi c})^{m}, \tag{4}$$

where A is now an empirical constant depending on the material, stress ratio, and crack geometry. This equation determines the number of cycles needed for each crack length to increase by Δc Substituting for the ΔN_i's in Equation 3, we finally obtain

$$L = \frac{1}{(\Delta\sigma)^{m}} \left(\frac{\Delta c}{A\,\pi^{\frac{m}{2}}} \right) \times \left(\frac{1}{c_0^{\frac{m}{2}}} + \cdots + \frac{1}{(c_0 + (M^{*}-1)\Delta c)^{\frac{m}{2}}} \right) . \tag{5}$$

Once we have a probability density function for the stress range, such as the commonly used Rayleigh distribution, the probability density function for L, is determined by a change of variables. This result is only useful as a consistency check. In particular, we know of no physical basis for the choice of the Rayleigh distribution for stress range.

4. Life Model Based on Stress-Intensity-Factor Range

We now derive a probability model consistent with Equation 1. Following Barsom [3], we assume that a representative value ΔK_μ of the stress-intensity-factor ranges can be used to express the crack-propagation behavior under a variable-amplitude load. Specifically, Barsom showed that the root mean square of the ΔK_i's may serve as such a representative value, i.e.,

$$\Delta K_\mu = \left(\frac{1}{N} \sum_{i=1}^{N} \Delta K_i^2 \right)^{\frac{1}{2}},$$

where N is the number of measurements until fatigue failure.

Conversely, when we have a value of ΔK_μ we can decompose it into ΔK_i's for $i = 1, 2, \cdots, N$. Here N is now the number of cycle blocks until fatigue failure. Each ΔK_i can be regarded as a random variable which determines the number of cycles for the ith unit increment of crack length.

Since ΔK_μ completely characterizes the fatigue process, the particular decomposition into the individual ΔK_i's should not influence the probability model. This implies the joint probability distribution for ΔK_i is invariant under transformations that leave ΔK_μ invariant. These transformations can be pictured as rotations in the N-dimensional space of ΔK_i's

[1, 9]. Distributions that are invariant under rotations are called rotationally-symmetric. In practice, the number N of cycle blocks is large. For large N, the rotationally-symmetric distributions tend to a conditional Gaussian probability distribution. In the present case, we find that the ΔK_i are each truncated Gaussian, conditional on ΔK_μ^2 which appears in the role of the traditional variance parameter:

$$f(\Delta k_i \mid \Delta k_\mu^2)\, d\Delta k_i \propto \frac{1}{\Delta k_\mu} \exp\left(-\frac{\Delta k_i^2}{2\,\Delta k_\mu^2} \right) d\Delta k_i \ . \tag{6}$$

The corresponding lower-case letters are being used for the realization of the upper-case random variables and the symbol " $\propto$" denotes "proportional to."

The difference equation corresponding to Equation 1 is

$$\frac{\Delta c}{\Delta N} = A(\Delta K)^m. \tag{7}$$

Here ΔK is generic for the ΔK_i Substituting the decomposition for L in Equation 3, we find for the relation between fatigue life L and ΔK

$$L = \frac{M^* \Delta c}{A} \frac{1}{(\Delta K)^m} \ .$$

A change of variable then gives the desired density for fatigue life:

$$f(l \mid \Delta k_\mu) dl \propto \left(\frac{1}{\Delta k_\mu}\right) l^{-\frac{1}{m}-1} \exp\left(-\frac{l^{-\frac{2}{m}}}{2\,\Delta k_\mu^2} \right) dl \ . \tag{8}$$

This distribution is not commonly used for fatigue life. However, the m^-th power of fatigue life L has a truncated Gaussian density function, which has been proposed as an alternative to the Birnbaum-Saunders model.

The expression in Equation 8 is a likelihood model for fatigue life. The model is Bayesian in the sense that the parameter ΔK_μ is a random variable having some prior distribution π. With a particular choice of prior, an unconditional or predictive distribution is determined as follows:

$$f(l)\, dl = \left[\int_0^\infty f(l \mid \Delta k_\mu)\, \pi(\Delta k_\mu)\, d\Delta k_\mu \right] dl \ . \tag{9}$$

5. Conclusions

The probability model for fatigue life presented here has two important distinguishing features. First, it is derived from the differential equations governing the fatigue process. Second, samples and other data are used only to update the fatigue life distribution (see Equation 9). These means that curve fitting techniques requiring large sample sizes can be avoided.

In practice, there are only small-sized samples available for predicting fatigue life data. Even though more accurate prediction can be expected from more samples, the proposed model can be utilized even in the case that only one sample is available. This is possible due to the fact that the model reflects the consistent behaviour of fatigue process. It doesn't rely on life data only. Life data are the sources for validating the probability model. It should not be a basis for the model. Therefore, the physical failure mechanism should be incorporated in the probability models.

The Weibull and the Birnbaum-Saunders distributions that are commonly used in the prediction of fatigue life are two-parameter families. The parameters in the Weibull and the Birnbaum-Saunders distributions are abstract statistical entities expressing mathematical properties of the probability distributions such as its shape and scaling [4, 11]. On the other hand, the distribution in Equation 8 has only a single parameter and this parameter has a direct physical meaning pertinent to the fatigue process (root mean squared stress-intensity-factor range). This implies that the probability distribution proposed here is "operational."

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