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M/G/1 Queueing System with Vacation and Limited-1 Service Policy

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ABSTRACT

In this paper we consider an M/G/1 queue where the server of the system has a vacation time and the service policy is limited-1. In this system, upon termination of a vacation the server returns to the queue and serves at most one message in the queue before taking another vacation. We consider two models. In the first, if the sever finds the queue empty at the end of a cacation, then the sever immediately takes another vacation. In the second model, if no message have arrived during a vacation, the sever waits for the first arrival to serve. The analysis of this system is particularly useful for a priority class polling system. We derive Laplace-Stieltjes transforms of the waiting time for both models, and compare their mean waiting times.

Keywords: M/G/1 queue, Poisson process, priority polling system, Laplace-Stieltjes transform.

1. Introduction

In an M/G/1 queueing system, it is assumed that the messages arrive at the queue according to a Poisson process, the service time has an arbitrary distribution and there is a single server. In this paper we consider an M/G/1 queue where the server of the system has a vacation time and the service policy is limited-1. In this syetem, the server has vacations and service periods alternately. Returning from a vacation, the server serves only one message, if any, in the queue and then takes another vacation. We consider two models. In the first, to be called *Model I*, when the server returns from a vacation and finds the system empty, it immediately takes another vacation, and continues in this manner until if finds at least one waiting message upon return from a vacation. The second model , to

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be called *Model II*, differs from Model I in that, when it finds the system empty at the end of a vacation, the server waits for the first message arrival to serve before taking another vacation.

The analysis of a queueing system with a vacation period is particularly useful in a priority polling system. A priority polling system, like a cyclic one, consists of a single server shared by multiple queues and each queue is served in an order specified in a so-called polling table. For a particular queue in this polling system, the server may be thought to have a vacation when it is serving the other stations. Thus, the results for a queueing system with a vacation period can be applied directly to a priority polling system. See, for example, Baker and Rubin (1987) and Ryu *et al.* (1998).

Eisenberg (1972), and Levy and Yechiali (1975) have considered an M/G/1 queueing system with a vacation period and exhaustive service policy. Here we complement their results in the case of limited-1 service. For Model I and II, we derive the Laplace-Stieltjes transform of the sojourn period which is the total time elapsing from the moment when a message arrives at the queue to the moment of service completion. From this we obtain the Laplace-Stieltjes transform of the waiting time for a message. The basic tool for doing this is to consider an embedded Markov chain defined on an extended state space. Furthermore, we calculate and compare the mean waiting times of the two models. It turns out that Model II is always more efficient than Model I.

2. Analysis of Model I

We suppose that the stream of message arrivals is a homogeneous Poisson process with rate λ. Let $N(t)$ denote the number of messages arriving at the queue during the time period t. Then, $N(t)$ follows a Poisson distribution with mean λt. Let S be the service time for a single message. Let V_0 denote a single vacation period. The random variables S and V_0 are allowed to have arbitrary distributions. Let V be the total period of consecutive single vacations.

We consider a Markov chain with transitions occurring at the times of service completion or vacation termination. Since these two instants activate different transition rules, we need to distinguish them in defining the state space of the numbers of messages in the queue. We define an extended state space $\{(i,j) : i = 0,1\,; j = 0,1,2,\ldots\} - \{(0,0)\}$ in such a way that if $i = 0$ then j counts the number of messages at the instant of vacation termination, and if $i = 1$ then j denotes the number of messages immediately after a service completion. Note here that the state $(0,0)$ does not exist in our model.

Let (i_n, j_n) be the state of the system at the n-th transition. The transition law for this semi-Markov chain is given by

$$(i_{n+1}, j_{n+1}) = \begin{cases} (1, j_n + N(S) - 1) & \text{if } i_n = 0,\ j_n \geq 1 \\ (0, j_n + N(V_0)) & \text{if } i_n = 1,\ j_n \geq 1 \\ (0, N(V)) & \text{if } i_n = 1,\ j_n = 0 \end{cases}.$$

Define $\pi_{ij} = \lim_{n\to\infty} \mathrm{P}(i_n = i,\ j_n = j)\,;\ i = 0,1,\ j = 0,1,2,\ldots$. These limiting probabilities satisfy the following equations :

$$\begin{aligned} \pi_{0j} &= \pi_{10}\mathrm{P}\{N(V) = j\} + \sum_{k=1}^{j} \pi_{1k}\mathrm{P}\{N(V_0) = j - k\},\ j = 1,2,\ldots \\ \pi_{1j} &= \sum_{k=1}^{j+1} \pi_{0k}\mathrm{P}\{N(S) = j - k + 1\},\ j = 0,1,2,\ldots.\} \end{aligned} \tag{2.1}$$

Write $c_0 = \mathrm{P}\{N(V_0) = 0\} = \mathrm{E}(e^{-\lambda V_0}) = L_{V_0}(\lambda)$. Here and below $L_X(\cdot)$ denotes the Laplace-Stieltjes transform of the distribution function of a random variable X. We have for $j \geq 1$

$$\begin{aligned} \mathrm{P}\{N(V) = j\} &= \mathrm{P}\{N(V) = j,\ N(V_0) = 0\} + \mathrm{P}\{N(V) = j,\ N(V_0) \geq 1\} \\ &= \mathrm{P}\{N(V_0) = j\}/(1 - c_0). \end{aligned} \tag{2.2}$$

We introduce the generating functions of $\{\pi_{0j}\}$ and $\{\pi_{1j}\}$. Define

$$\pi_0(z) = \sum_{j=1}^{\infty} z^j \pi_{0j}, \quad \pi_1(z) = \sum_{j=0}^{\infty} z^j \pi_{1j}.$$

From the equations (2.1) and (2.2), we obtain

$$\begin{aligned} \pi_0(z) &= \sum_{j=1}^{\infty} z^j \left\{ \frac{\pi_{10}}{1 - c_0} \mathrm{P}\left(N(V_0) = j\right) + \sum_{k=1}^{j} \pi_{1k}\mathrm{P}\left(N(V_0) = j - k\right) \right\} \\ &= \sum_{j=0}^{\infty} z^j \left\{ \frac{c_0\pi_{10}}{1 - c_0} \mathrm{P}\left(N(V_0) = j\right) + \sum_{k=0}^{j} \pi_{1k}\mathrm{P}\left(N(V_0) = j - k\right) \right\} \\ &\quad - \frac{\pi_{10}}{1 - c_0} \mathrm{P}\left\{N(V_0) = 0\right\} \end{aligned}$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \pi_{1k} z^k \sum_{\ell=0}^{\infty} z^\ell \mathrm{P}\left\{N(V_0) = \ell\right\} + \frac{c_0 \pi_{10}}{1-c_0} \sum_{j=0}^{\infty} z^j \mathrm{P}\left\{N(V_0) = j\right\} \\ &\quad - \frac{\pi_{10}}{1-c_0} \mathrm{P}\left\{N(V_0) = 0\right\} \\ &= \pi_1(z) G_{N(V_0)}(z) - \frac{c_0 \pi_{10}}{1-c_0}\left\{1 - G_{N(V_0)}(z)\right\}. \end{aligned} \tag{2.3}$$

Similarly we get

$$\pi_1(z) = z^{-1} \pi_0(z) G_{N(S)}(z). \tag{2.4}$$

From the equations (2.3) and (2.4), we obtain

$$\pi_1(z) = \frac{c_0 \pi_{10}\left\{1 - G_{N(V_0)}(z)\right\} G_{N(S)}(z)}{(1-c_0)\left\{G_{N(V_0)}(z) G_{N(S)}(z) - z\right\}}. \tag{2.5}$$

By applying l'Hôpital's rule on (2.5), we may obtain

$$\pi_1(1) = \frac{c_0\, \pi_{10}\, \lambda\, \mathrm{E}(V_0)}{(1-c_0)\left\{1 - \lambda \mathrm{E}(V_0) - \lambda \mathrm{E}(S)\right\}}.$$

We now derive the Laplace-Stieltjes transform of the sojourn period, denoted by C, and the waiting time, denoted by W, of a message. Note that $\mathrm{P}\{N(C) = j\} = \pi_{1j}/\pi_1(1)$, hence

$$G_{N(C)}(z) = \mathrm{E}(z^{N(C)}) = \pi_1(z)/\pi_1(1).$$

Since $G_{N(C)}(z) = L_C(\lambda(1-z))$, we obtain from (2.5) that

$$L_C(\lambda(1-z)) = \frac{\left\{1 - \lambda \mathrm{E}(V_0) - \lambda \mathrm{E}(S)\right\}\left\{1 - L_{V_0}(\lambda(1-z))\right\} L_S(\lambda(1-z))}{\lambda\, \mathrm{E}(V_0)\left\{L_{V_0}(\lambda(1-z)) L_S(\lambda(1-z)) - z\right\}}.$$

Thus, by letting $\alpha = \lambda(1-z)$, we get

$$L_C(\alpha) = \frac{\left\{1 - \lambda \mathrm{E}(V_0) - \lambda \mathrm{E}(S)\right\}\left\{1 - L_{V_0}(\alpha)\right\} L_S(\alpha)}{\lambda\, \mathrm{E}(V_0)\left\{L_{V_0}(\alpha) L_S(\alpha) - (1 - \alpha/\lambda)\right\}}.$$

Since $C = W + S$ and the two random variables W and S are independent, we have $L_W(\alpha) = L_C(\alpha)/L_S(\alpha)$. Thus

$$L_W(\alpha) = \frac{\left\{1 - \lambda \mathrm{E}(V_0) - \lambda \mathrm{E}(S)\right\}\left\{1 - L_{V_0}(\alpha)\right\}}{\lambda\, \mathrm{E}(V_0)\left\{L_{V_0}(\alpha) L_S(\alpha) - (1 - \alpha/\lambda)\right\}}. \tag{2.6}$$

3. Analysis of Model II

Here again we consider a Markov chain with transitions occuring at the epochs of service completion or vacation termination. We note that in this model the state space includes the state (0,0), i.e. the instant where the server finds the queue empty at the end of a vacation. The transition law for the corresponding semi-Markov chain is given by

$$(i_{n+1}, j_{n+1}) = \begin{cases} (1, j_n + N(S) - 1) & \text{if } i_n = 0,\ j_n \geq 1 \\ (1, N(S)) & \text{if } i_n = 0,\ j_n = 0 \\ (0, j_n + N(V_0)) & \text{if } i_n = 1. \end{cases}$$

The limiting state probabilites then satisfy the following equations :

$$\begin{aligned} \pi_{0j} &= \sum_{k=0}^{j} \pi_{1k} \mathrm{P}\{N(V_0) = j - k\},\ j = 0, 1, 2, \ldots \\ \pi_{1j} &= \pi_{00} P\{N(S) = j\} + \sum_{k=1}^{j+1} \pi_{0k} \mathrm{P}\{N(S) = j - k + 1\},\ j = 0, 1, 2, \ldots. \end{aligned} \tag{3.1}$$

From (3.1) we can show the generating functions $\pi_0 = \sum_{j=0}^{\infty} z^j \pi_{0j}$ and $\pi_1(z) = \sum_{j=0}^{\infty} z^j \pi_1 j$ satisfy the following two equations :

$$\begin{aligned} \pi_0(z) &= \pi_1(z) G_{N(V_0)}(z) \\ \pi_1(z) &= \{z^{-1}\pi_0(z) + (1 - z^{-1})\pi_{00}\} G_{N(S)}(z). \end{aligned} \tag{3.2}$$

From the equation at (3.2), we obtain

$$\pi_1(z) = \frac{(1 - z^{-1}) G_{N(S)}(z) \pi_{00}}{\{1 - z^{-1} G_{N(S)}(z) G_{N(V_0)}(z)\}}. \tag{3.3}$$

Applying l'Hôpital's rule on (3.3), we get $\pi_1(1) = \{1 - \lambda \mathrm{E}(V_0) - \lambda \mathrm{E}(S)\}^{-1} \pi_{00}$. The Laplace-Stieltjes transform of the waiting time W is derived in a way similar to the derivation of (2.6) and is

$$L_W(\alpha) = \frac{\alpha\{1 - \lambda \mathrm{E}(V_0) - \lambda \mathrm{E}(S)\}}{\{\alpha + \lambda L_S(\alpha) L_{V_0}(\alpha) - \lambda\}}. \tag{3.4}$$

4. Comparison Between the Models

We compare the mean waiting times of a message. First, we derive the formula for the mean waiting time in each of the two models. For Model I, it follows from (2.6) that the mean waiting time, denoted by $\mathrm{E}(W_I)$, is given by

$$\begin{aligned} \mathrm{E}(W_I) &= -L'_W(0) \\ &= \frac{1}{2}\frac{\mathrm{E}(V_0^2)}{\mathrm{E}(V_0)} + \frac{\lambda}{2}\frac{\mathrm{E}(V_0^2) + 2\mathrm{E}(V_0)\mathrm{E}(S) + \mathrm{E}(S^2)}{1 - \lambda\mathrm{E}(V_0) - \lambda\mathrm{E}(S)}. \end{aligned} \tag{4.1}$$

The equation (4.1) may be verified by applying l'Hôpital's rule twice. The mean waiting time for Model II, denoted by $\mathrm{E}(W_{II})$, is also obtained by applying l'Hôpital's rule successively on (3.4) and is given by

$$\mathrm{E}(W_{II}) = \frac{\lambda}{2}\frac{\mathrm{E}(V_0^2) + 2\mathrm{E}(V_0)\mathrm{E}(S) + \mathrm{E}(S^2)}{1 - \lambda\mathrm{E}(V_0) - \lambda\mathrm{E}(S)}. \tag{4.2}$$

From (4.1) and (4.2), the mean waiting time for Model II is always smaller than the one for Model I by the amount $\mathrm{E}(V_0^2)/\{2\,\mathrm{E}(V_0)\}$. In the case, for example, where a single vacation period V_0 follows an exponential distribution with mean μ, this equals μ. Thus, in this case, the difference between the mean waiting times is amplified as the average single vacation period increases.

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