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Realistic Equivalent Load Methods in Prestressed Concrete Structures

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Abstract

The purpose of this state-of-the-art paper is to explore several important methods for obtaining the equivalent loads in prestressed concrete structures, and to clarify the theoretical basis and implied assumptions of each method. The methods devised in this study include the use of curvature of tendon, characteristics of primary moment, self-equilibrium condition and linear segments approximation of tendon. It is shown that equivalent loading system is not uniquely determined depending on the approach adopted to calculate the equivalent loads. Self-equilibrium conditions of the equivalent loading system are carefully discussed. Numerical examples are presented to show the differences among the methods and results of the approximations in each method explicitly.

keywords : equivalent load method, prestressed concrete structure, prestressing tendon, self-equilibrium condition

1. Introduction

With the progress of analysis technique for prestressed concrete structures, many studies to investigate the effect of prestressing tendon on the structural behavior have been conducted.

Concerning the modeling of tendon in prestressed concrete structures, two alternative approaches are generally used[1]. One is to deal with tendon as load resisting elements in the finite element analysis of prestressed concrete structures. The other is to evaluate only the forces exerted to structure by prestressing tendon, neglecting stiffness of tendon.

The first approach, which regards tendon as load resisting elements, is also similarly used to model the reinforcing steel bar in the finite element analysis of reinforced concrete structures, where the difference between tendon and steel bar is the existence of initial stress. This approach can consider not only the stiffness of tendon but also the effect of change of prestressing force according to the deformation of a structure.

The second approach can be classified into several categories, the representative of which is the equivalent load method that considers prestressing effect exerted to the structure as equivalent external load. By employing this concept, one can obtain a clear picture of the forces acting in the prestressed concrete structures and it allows the designer to use familiar beam and frame analysis. That is, the analysis of prestressed concrete member is converted to that of nonprestressed member.

The equivalent load method has wide applications in the design and analysis of prestressed concrete structures. The examples are flexural analysis utilizing load balancing method[2], shear analysis[3], and assessment of friction loss[4] etc. Furthermore, the equivalent load method can be used to analyze not only various kinds of determinate or indeterminate frame structures including simple beams, cantilevers, continuous beams, and rigid frames, but also two or three dimensional structures such as slabs, grid systems, and shell structures[5]. Aalami[1,6] studied equivalent load method for the case where centroidal axis abruptly changes. As was mentioned above, equivalent load method is simple yet powerful one for analyzing prestressed concrete structures, and is preferred in the design and analysis carried out by hand calculations and conventional frame analysis program without special provisions for tendon.

In the light of aesthetics of structures, the complex shaped prestressed concrete members including bent or curved members, and the members with varying cross-section, etc, will be constructed more frequently. For the equivalent load method to be the standard design and analysis method of various kinds of prestressed concrete members, it is however required that rational methodology should be established and limitations specified, because the equivalent load method in many cases is approximate for computational convenience.

The main purpose of the present study is, therefore, to devise several important methods of calculating equivalent loads, and to clarify the advantages and limitations of various methods studied. Some equivalent loading systems are presented to show possible approximations of equivalent load method and the results which originate from those approximations.

2. Derivation of the Equivalent Loads

The equivalent loads consist of distributed and/or concentrated loads inside the member and concentrated loads at the end anchorages. Some important methods and procedures used to find equivalent loads will be devised and thoroughly examined in this section and the approximations used in each method will also be discussed. Constant prestressing force is assumed in each method, however immediate and long term stress losses in prestressing can also be approximated in some methods[1].

2.1 Method 1: Equivalent Load Calculated by Curvature of the Tendon

The original equivalent load depends only on the geometry of tendon, precisely speaking, curvature of tendon irrespective of shape of the member and support condition. The distributed load caused by the curvature of tendon at the contact face between tendon and surrounding concrete can be written as[7],

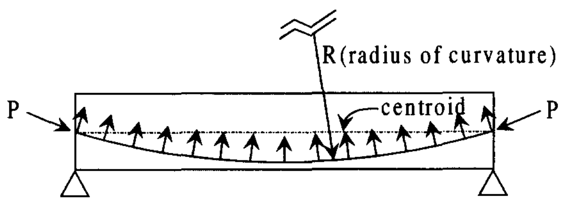
$$w = \frac{P}{R} = \kappa P \tag{1}$$

with the direction normal to the tendon axis. Here, P is the prestressing force, R is the radius of curvature, and κ is the curvature. For a curve $y = y(x)$ in xy-plane[8],

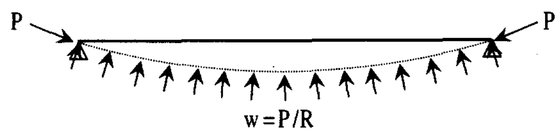
$$\kappa(x) = \frac{|y''|}{(1+y'^2)^{3/2}} \tag{2}$$

This method is convenient to use for the tendon with constant curvature as in the circular tendon profile. Fig. 1(a) shows the exact equivalent loads in this case for the simple beam. However, the problem is now how one can accommodate these equivalent loads in the beam analysis. The best answer to this is to deal with equivalent loads as they are without any modification in the magnitude and direction already calculated. The idealized beam is represented by a line which usually indicates centroid of the member. Fig. 1(b) shows the idealized simple beam in question with exact equivalent loads applied. As can be expected, beam analysis of Fig. 1(b) results in exact section forces. These exact section forces are derived in Appendix.

However, in many cases including above example, the equivalent load method is not used in a strict sense due to the practical considerations for the convenience of structural analysis. That is, applied location, direction and magnitude of the equivalent distributed load which are originally calculated by curvature of the tendon along the tendon axis are modified depending on the cases. This is referred to as the method 1 in an approximate sense.



(a) Equivalent loads in a strict sense



(b) Idealized beam with exact equivalent loads

Fig. 1 Simple beam with circular tendon profile

Fig. 2 is a generally accepted analysis model of Fig. 1(b) where original equivalent distributed load is transferred to the location of centroidal axis of the member with its magnitude maintained but direction slightly changed. These approximations result in the following problems.

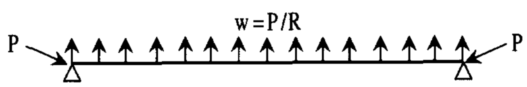


Fig. 2 Approximate equivalent loading system for the simple beam of Fig. 1(b)

Firstly, the equivalent loads do not satisfy self-equilibrium condition, which must be met from the consideration of statics. As a result, some reactions of small magnitude at the supports can occur, which do not exist in the simple prestressed beam in reality. This problem will be discussed in the following section in more detail. Secondly, the section forces produced by approximate equivalent loads are also approximate ones.

As is seen above, method 1 can further be classified into two methods, exact one or approximate one, which will be separately specified hereafter for distinction.

2.2 Method 2: Equivalent Load Calculated by the Primary Moment

When the curvature of tendon varies from point to point, the method 1 is not practical and the following method can be used instead especially for the tendon of polynomial type. The primary moment can be defined as follow for small value of θ.

$$M_P = (P\cos\theta)e \approx Pe \tag{3}$$

This is very effective assumption in the sense that the bending moment diagram of M_P has the same shape as eccentricity with only difference of magnitude. The procedure for obtaining the equivalent load from the primary moment can be summarized as follows[7].

(1) Plot the primary moment diagram $M_P(x) = Pe$ throughout spans.
(2) From the primary moment diagram, determine the corresponding shear diagram either graphically or algebraically.
(Note: $V(x) = dM_P(x)/dx = Pe'$)
(3) From the shear diagram, determine the equivalent loading diagram.
(Note: $w(x) = -dV(x)/dx = -Pe''$)

Note that according to this procedure, the parabolic eccentricity corresponds to uniform equivalent load, and the cubic eccentricity to linearly varying one, and so on. In step 3, one has implicitly accepted approximate expression of the curvature, i.e., the numerator part of Eq. (2), since small θ assumption means $y' = 0$. Therefore, by the above procedure parabolic tendon can be treated as the equivalent circular tendon with constant curvature demonstrated in the method 1.

Consider the simple beam with parabolic tendon profile. With the coordinate definition shown in Fig. 3, the eccentricity can be written as $e = 4hx(1 - x/L)/L$. Proceeding from step 1 to step 3 results in $w = 8Ph/L^2$. This produces exact bending moment at the mid-span, but approximate one at other locations. This is due to the fact that $M_P = Pe$ is true expression at the mid-span and approximate one at other locations.

Next procedure can also be used for the parabolic tendon which is based on the above discussions. It is assumed that the equivalent load is applied at the centroidal axis of a beam with constant magnitude in the vertical direction, although exact equivalent load is applied along the tendon axis and the magnitude slightly varies from point to point because the curvature is not constant for parabola. Then, the magnitude is determined so that it produces exact bending moment at the mid-span.

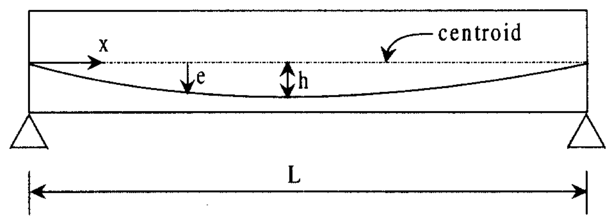


Fig. 3 Simple beam with parabolic tendon profile

Self-equilibrium condition and section forces produced by the method 2 are only approximate expressions, because the above procedure has the same effect as slightly modifying the magnitude and direction of exact equivalent load.

As a comparison, the primary moment diagram itself can also be used for the analysis instead of converting to the equivalent distributed load[1]. In this method, the primary moment diagram is discretized into a number of steps and a number of concentrated moments are applied to the member which produce the discretized primary moment diagram Irrespective of the loss of accuracy from the discretization, this method is widely used from the practical consideration. Since exact mathematical expression of eccentricity for differentiation is not required, this method can easily be applied to any tendon profile and cross-sectional geometry of the member.

2.3 Method 3: Equivalent Load Calculated by the Self-Equilibrium Condition

This approach evaluates the equivalent load by equilibrium condition with anchorage forces given. Therefore, the self-equilibrium condition, which is satisfied in an approximate sense in the methods 1 and 2, is exactly satisfied, although section forces are still approximate values.

Two equilibrium equations can be used to find equivalent vertical load in the plane beam, i.e., vertical force equilibrium and moment equilibrium. One should try to find out the equivalent loads that satisfy these two equilibrium equations with the anchorage forces given at both ends of the beam. This may raise a question, because there exist many load cases that satisfy equilibrium condition with anchorage forces. Therefore, it is necessary that one should obtain some information of equivalent loads from the geometry of tendon with two unknowns left before detailed calculations.

Consider the member with cubic tendon profile and anchorage forces given in Fig. 4(a). It is already known that assuming constant horizontal prestressing force, the cubic tendon profile corresponds to linearly varying equivalent load from the method 2. Therefore, it is reasonable to set the magnitudes of linearly varying load at both ends of the member as unknowns, and solve for these two unknowns by two equilibrium equations. Referring to Fig. 4(b) with positive sign conventions shown, the following results are derived from the equilibrium conditions. These equations are used, for example, in Ref. 9.

$$w_i = -\frac{2}{L}(2P\sin\theta_i - P\sin\theta_j) - \frac{6}{L^2}(e_iP\cos\theta_i + e_jP\cos\theta_j) \tag{4}$$

$$w_j = \frac{2}{L}(P\sin\theta_i - 2P\sin\theta_j) + \frac{6}{L^2}(e_iP\cos\theta_i + e_jP\cos\theta_j) \tag{5}$$

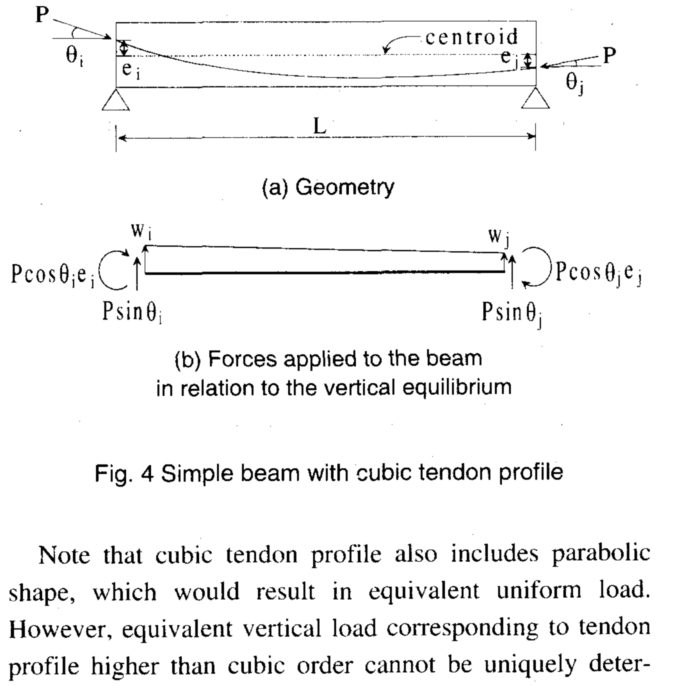


Fig. 4 Simple beam with cubic tendon profile

Note that cubic tendon profile also includes parabolic shape, which would result in equivalent uniform load. However, equivalent vertical load corresponding to tendon profile higher than cubic order cannot be uniquely determined by only two equilibrium equations, because the number of unknowns exceeds two.

When the several kinds of profiles are used in one tendon as in the continuous beam, the beam should be divided into segments so that each segment has the part of tendon with unique polynomial expression. After that, the method 3 can be applied to each segment to find corresponding equivalent load, respectively.

2.4 Method 4: Equivalent Load by Linear Segments Approximation of Tendon

According to the method 4, the member is divided into segments and the curved tendon is assumed to be straight tendon connecting two intersection points in each segment. Then, the equivalent load exists as the form of concentrated load only at the interface of two segments meeting each other and the equivalent distributed load arising from the curvature of tendon disappears. Method 4 is the frequently adopted scheme in many studies[1,10,11].

As can be expected, to simulate curved geometry of tendon by this method to some extent, many segments are required depending on the problem.

3. Discussion on the Self-Equilibrium Condition

3.1 Equilibrium in Vertical Direction

As mentioned earlier, the equivalent loading system should be in self-equilibrium condition, so that it causes no reaction at the supports for determinate structure. However, it was already pointed out that the self-equilibrium is satisfied in an approximate sense for the methods 1 (approximate) and 2, because the equivalent load is not determined taking equilibrium into account in these methods. On the other hand, in the method 3 which starts from the equilibrium consideration, the self-equilibrium in vertical direction is naturally satisfied. One can also find that the self-equilibrium is still maintained in the linear segments approximation of the method 4.

Fig. 5 shows two approximate equivalent loading systems with parabolic tendon profile. In the hand calculation, one may calculate to obtain section forces as if there is no support, judging that the system is in self-equilibrium and causes no reaction. Note that both the members in Fig. 5 are simply supported and thus reactions are not caused by the deformation of member. However, slight discrepancies can be introduced in this procedure when the equivalent load is determined by the method 1(approximate) or 2, because the self-equilibrium in vertical direction is not satisfied exactly.

Fig. 6 represents vertical reactions of each beam calculated from the equilibrium condition. For the method 3, these reactions are zero, while reactions of small magnitude exist for the equivalent load w according to the method 1(approximate) or 2. In the structural analysis program, these reactions are naturally introduced to the system, because the system finds its equilibrium condition with these reactions.

One should note that this kind of 'parasitic reaction' could produce slightly different section forces compared to what may be expected. All these situations arise from the approximation used for the equivalent load.

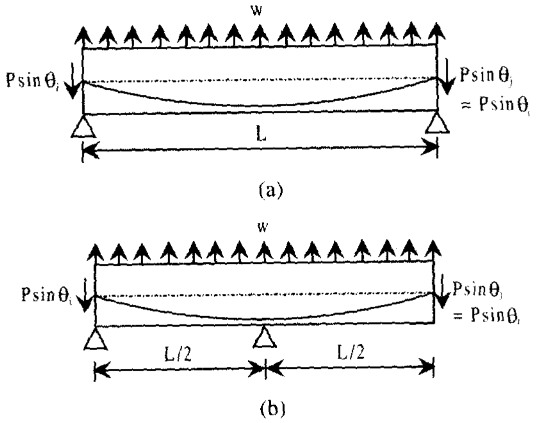


Fig. 5 Two examples of equivalent vertical loading system with parabolic tendon profile

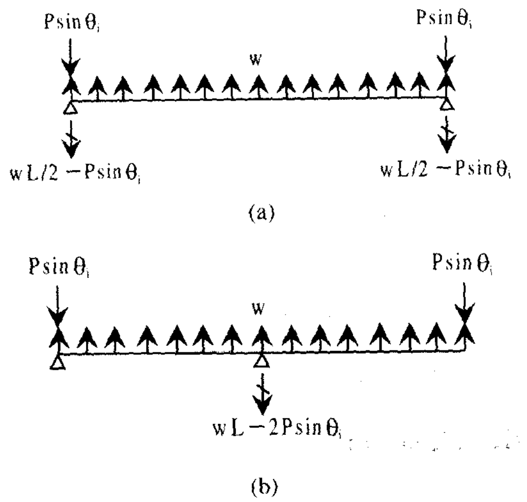


Fig. 6 Parasitic vertical reactions arising in the examples of Fig. 5

3.2 Equilibrium in Axial Direction

Self-equilibrium condition should also be satisfied in axial direction. However, when we note that the methods 1(approximate), 2 and 3 are oriented for equivalent 'vertical' load with the effect of axial load due to prestressing completely decoupled, there is no special consideration on the axial equilibrium during this decoupling process. It can be readily seen that self-equilibrium in axial direction is not satisfied by these methods when the axial components of anchorage forces at both the ends of beam have different values. For example, if the axial component of anchorage force is not assumed to be P in Fig. 4(a) differently from the frequently adopted assumption, parasitic axial reaction is induced as in Fig. 7, since the system is not self-equilibrated.

On the other hand, self-equilibrium in axial direction is maintained in the methods 1(exact) and 4.

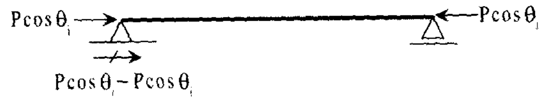


Fig. 7 Parasitic axial reaction arising in the example of Fig. 4(a)

4. Numerical Example

A simple beam with parabolic tendon profile shown in Fig. 8 is analyzed. The beam has a span of 8m with eccentricity of 25cm at the mid-span. Fig. 9 illustrates the equivalent loading systems set up by each method.

In the method 1, the equivalent load is calculated by Eqs. (1) and (2). Applied locations, directions and magnitudes of so calculated equivalent loads are not approximated here, that is, those are taken into account in an exact sense here. Therefore, the solution from method 1 is exact in this case.

For the method 2, the equivalent load is obtained by $w = 8Ph / L^2$ based on the procedures presented previously. Parasitic vertical reactions induced by the method 2 should be noticed in the Fig. 9(b). Therefore, one should note that concentrated vertical load at the ends of the beam in this case is not purely the vertical component of the anchorage force itself, which is 124.04 kN, but is calculated by 124.04 kN + 0.96 kN = 125 kN as a result of these reactions.

Equilibrium considerations of Eqs. (4) and (5) are used in the method 3, which produces rather different magnitude of equivalent load compared to that of method 2. Of course, no reaction is induced by the self-equilibrium considerations adopted for method 3 in the case of determinate structures including present simple beam.

Accuracy of the method 4 is strongly dependent upon the degree of linear segments approximation. Here, the tendon is discretized into four linear segments.

Fig. 10 shows bending moment diagrams produced by each equivalent loading system, which are calculated by ordinary beam analysis. The value of moment is negative at all points of the beam. Bending moments at x=L/4 and x=L/2 are described in each diagram. The shape of bending moment diagram is approximate parabola for Fig. 10(a) and exact parabola for Figs. 10(b) and 10(c). As proved in the Appendix, exact section forces are produced by the equivalent loading system of method 1 in an exact sense.

In the Fig. 10(b) produced by method 2, the bending moment at x=L/4 is slightly approximate although exact bending moment is derived at the mid-span. This is peculiar aspect of method 2 depending on the validity of approximation of Eq. (3) as pointed out already.

For the Fig. 10(d), discontinuities of bending moment exist at some bent points of linearized tendon, i.e., at x=L/4 and x=3L/4 in the present case. This results from the intermediate horizontal forces of magnitude 3.88 kN, which are applied due to the slope discontinuities of linearly idealized tendon at these points.

The present example clearly shows the special characteristics of each equivalent load method. One can capture the deviations of resulting bending moments from the exact solution and also can distinguish the accuracy and applicability of each method. This is the main feature of the present paper.

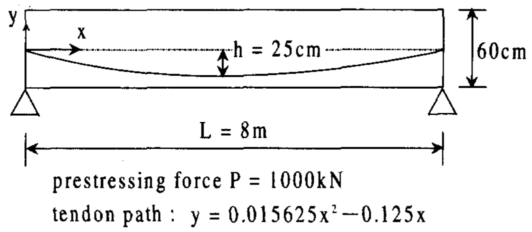
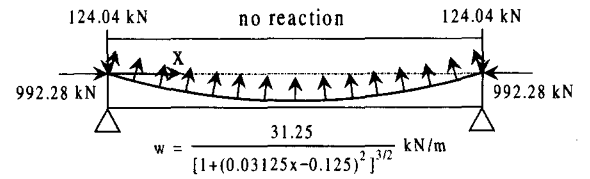
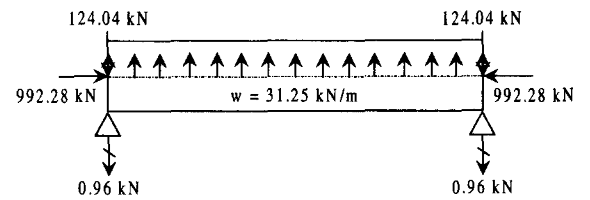


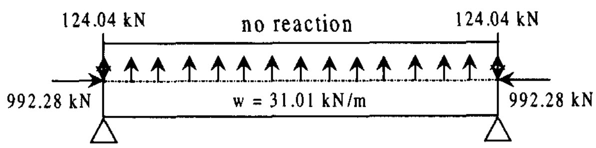
Fig. 8 Numerical example



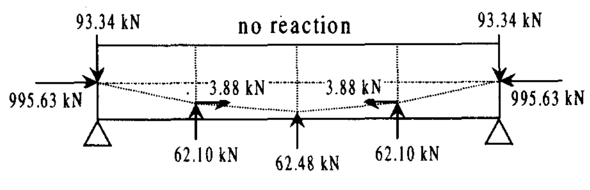
(a) Method 1(exact)



(b) Method 2

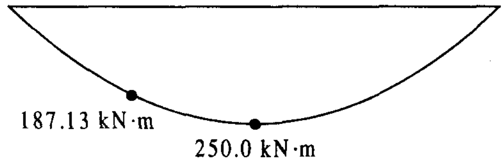


(c) Method 3

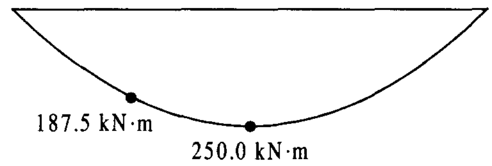


(d) Method 4

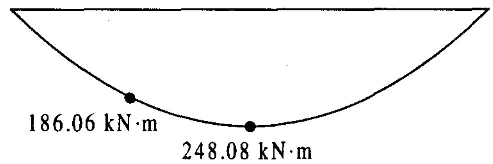
Fig. 9 Equivalent loading systems



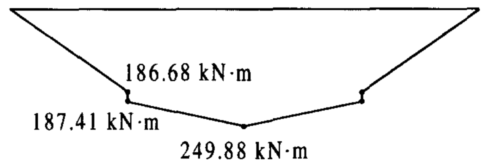
(a) Method 1(exact)



(b) Method 2



(c) Method 3



(d) Method 4

Fig. 10 Bending moment diagrams

5. Conclusions

(1) Several important methods of obtaining equivalent loads have been investigated and compared each other. Theoretical basis and implications of each method are emphasized. The methods devised in this study include the use of curvature of tendon, characteristics of primary moment, self-equilibrium condition and linear segments approximation of tendon.

(2) The approximations of each equivalent load method and the results which originate from those approximations have been explicitly demonstrated through theoretical and numerical approaches. Different equivalent loading system and resulting section forces can be produced even from the same prestressed concrete structure depending on the method.

(3) Useful discussions on the self-equilibrium conditions of equivalent loading system have been presented, which indicate that some parasitic reactions in vertical and axial directions may arise by some equivalent loads even for the determinate structures.

(4) Through careful examinations for obtaining the equivalent loads, it is expected that the equivalent load method

can be applied to various kinds of prestressed concrete members in a more rational and efficient manner.

Notation

w = equivalent distributed load by tendon
P = prestressing force
R = radius of curvature of tendon
κ = curvature of tendon
M_P = primary moment
θ = angle between member axis and tangent to tendon axis
e = eccentricity of tendon
h = eccentricity of tendon at mid-span
L = length of member

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APPENDIX: Derivation of the Section Forces Produced by the Exact Equivalent Loads for the Circular Tendon Profile

Referring to Fig. A1, the section forces at a distance x from left end of the beam are calculated as follows. All results coincide with exact values calculated from the components of prestressing force and eccentricity.

Axial force:

$$\begin{aligned} N &= P\cos\theta_{x=0} + \int_{\theta_x}^{\theta_{x=0}} w\sin\theta R d\theta \\ &= P\cos\theta_{x=0} - wR\cos\theta\,\Big|_{\cos\theta_x}^{\cos\theta_{x=0}} = P\cos\theta_x \end{aligned} \quad \text{(A1)}$$

Shear force:

$$\begin{aligned} V &= P\sin\theta_{x=0} - \int_{\theta_x}^{\theta_{x=0}} w\cos\theta R d\theta \\ &= P\sin\theta_{x=0} - wR\sin\theta\,\Big|_{\sin\theta_x}^{\sin\theta_{x=0}} = P\sin\theta_x \end{aligned} \quad \text{(A2)}$$

Bending moment:

$$\begin{aligned} M &= P\sin\theta_{x=0}x - \int_{\theta_x}^{\theta_{x=0}} w\cos\theta(R\sin\theta - R\sin\theta_x)Rd\theta \\ &\quad + \int_{\theta_x}^{\theta_{x=0}} w\sin\theta(R\cos\theta - R\cos\theta_{x=0})Rd\theta \\ &= P\sin\theta_{x=0}(R\sin\theta_{x=0} - R\sin\theta_x) + \int_{\theta_x}^{\theta_{x=0}} PR\cos\theta\sin\theta_x d\theta \\ &\quad - \int_{\theta_x}^{\theta_{x=0}} PR\sin\theta\cos\theta_{x=0}d\theta \\ &= PR(\sin\theta_{x=0})^2 - PR\sin\theta_x\sin\theta_{x=0} + PR\sin\theta_x\sin\theta\,\Big|_{\sin\theta_x}^{\sin\theta_{x=0}} \\ &\quad + PR\cos\theta_{x=0}\cos\theta\,\Big|_{\cos\theta_x}^{\cos\theta_{x=0}} \\ &= P\cos\theta_x(R\cos\theta_x - R\cos\theta_{x=0}) = P\cos\theta_x\, e \end{aligned} \quad \text{(A3)}$$

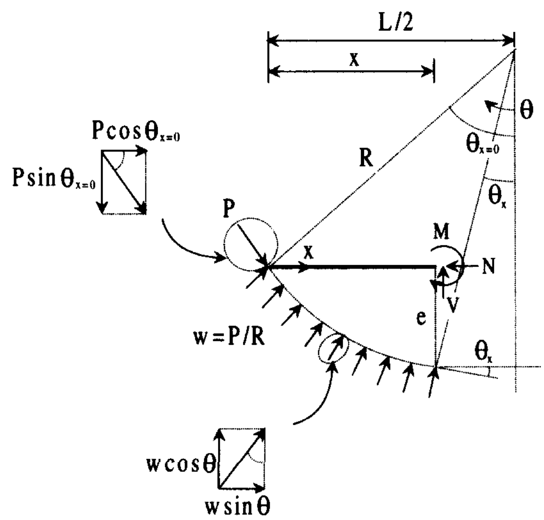

Fig. A1 Section forces of the beam with circular tendon profile