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ON FUZZY QUASI-CONTINUOUS MAPPINGS

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Abstract

The aim of this paper is to continue the study of fuzzy quasi-continuous mappings due to Park et al. [12] on fuzzy bitopological spaces.

Key Words : Fuzzy bitopological spaces, fuzzy quasi-continuous mappings

1. Introduction and preliminaries

Chang [2] used the concept of fuzzy sets to introduce fuzzy topological spaces and several authors continued the investigation of such spaces.

From the fact that there are some non-symmetric fuzzy topological structures, Kubiak [8] first introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological space, and initiated the bitopological aspects due to Kelly [7] in the theory of fuzzy topological spaces. Since then several authors [3,4,6,8-10,12] have contributed to the subsequent development of various fuzzy bitopological properties. Recently, Park et al. [12] defined and studied fuzzy quasi-open sets and fuzzy quasi-continuous mappings on fuzzy bitopological spaces. The aim of this paper is to continue the study of fuzzy quasicontinuous mappings between fuzzy bitopological spaces.

For definitions and results not explained in this paper, we refer to the papers [2,11-13] assuming them to be well known. A fuzzy point in X with support $x\in X$ and value α $(0<\alpha\leq 1)$ is denoted by x_α. For a fuzzy set A of $X, 1-A$ will stand for the complement of A. By 0_X and 1_X we will mean respec- tively the constant fuzzy sets taking on the values 0 and 1 on X.

A system (X,τ_1,τ_2) consisting of a set X with two topologies τ_1 and τ_2 on X is called a fuzzy bitopological space [7] (for short, fbts). A fuzzy set A of a fbts (X,τ_1,τ_2) is called fuzzy quasi-open [12] (briefly, fqo) if for each fuzzy point $x_\alpha\in A$ there is either a $U\in\tau_1$ such that $x_\alpha\in U\leq A$, or a $V\in\tau_2$ such that $x_\alpha\in V\leq A$. A fuzzy set A is fuzzy quasi-closed (briefly, fqc) if the complement $1-A$ is a fqo set. A fuzzy set A of a fbts (X,τ_1,τ_2) is called a quasi-Q-nbd [12] (resp. quasi-nbd [12]) of a fuzzy point x_α if there exists a fqo set U such that $x_\alpha \, q \, U\leq A$ (resp. $x_\alpha\in U\leq A$).

Result 1 [12]. A fuzzy set A of a fbts (X,τ_1,τ_2) is fqo if and only if there exist $U\in\tau_1$ and $V\in\tau_2$ such that $A=U\cup V$.

Result 2 [12]. Let A be any fuzzy set of a fbts X. Then $x_\alpha\in \mathrm{qcl}(A)$ if and only if for each fqo quasi-Q-nbd U of x_α, $U q A$.

2. Some properties of fuzzy quasi-continuous mappings

A mapping $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be fuzzy quasi-continuous [12] if $f^{-1}(B)$ is fqo in X for each $B\in\sigma_i$, equivalently, $f^{-1}(B)$ is fqc in X for each σ_i-fc set B of Y.

Theorem 2.1. For a mapping $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following are equivalent:

(a) f is fuzzy quasi-continuous.

(b) $f^{-1}(V)$ is fqo in X for each fqo set V of Y.

Proof. (a)$\Rightarrow$(b): Let V be any fqo set of Y. By result 1, there exist $V_1\in\sigma_1$ and $V_2\in\sigma_2$ such that $V=V_1\cup V_2$. Since f is fuzzy quasi-continuous, $f^{-1}(V)=f^{-1}(V_1\cup V_2)=f^{-1}(V_1)\cup f^{-1}(V_2)$ is fqo in X.

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(b) $\Rightarrow$ (a): Let $V \in \sigma_i$. Since every σ_i-fuzzy open set is fqo, $f^{-1}(V)$ is fqo in X. Hence f is fuzzy quasi-continuous.

Theorem 2.2. For a mapping $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following are equivalent:

(a) f is fuzzy quasi-continuous.

(b) For each fuzzy point x_α in X and each fqo quasi-nbd V of $f(x_\alpha)$, there is a fqo quasi-nbd U of x_α such that $f(U)\le V$.

(c) For each fuzzy point x_α in X and each quasi-Q-nbd V of $f(x_\alpha)$, there is a quasi-Q-nbd U of x_α such that $f(U)\le V$.

(d) $f(\mathrm{qcl}(A))\le \mathrm{qcl}(f(A))$ for each fuzzy set A of X.

(e) $\mathrm{qcl}(f^{-1}(B))\le f^{-1}(\mathrm{qcl}(B))$ for each fuzzy set B of Y.

Proof. (a) $\Rightarrow$ (b): Let x_α be any fuzzy point in X and V be any fqo quasi-nbd of $f(x_\alpha)$. Then $f^{-1}(V)=U$ (say) is a fqo quasi-nbd of x_α such that $f(U)\le V$.

(b) $\Rightarrow$ (c): Let x_α be any fuzzy point in X and V be any fqo quasi-Q-nbd of $f(x_\alpha)$. Since $V(f(x))+\alpha>1$, there exists a real number $\beta>0$ such that $V(f(x))>\beta>1-\alpha$, so that V is fqo quasi-nbd of $f(x)_\beta$. By (b), there is a fqo quasi-nbd U of x_β such that $f(U)\le V$. Now, $U(x)\ge\beta$ implies $U(x)>1-\alpha$ and thus U is a fqo quasi-Q-nbd of x_α.

(c) $\Rightarrow$ (d): Suppose that $x_\alpha\in \mathrm{qcl}(A)$ such that $f(x_\alpha)\notin \mathrm{qcl}(f(A))$. Then there is a fqo quasi- Q-nbd V of $f(x_\alpha)$ such that $V \not\mathrm{q} f(A)$ which implies $A\le 1-f^{-1}(V)$. By (c), there exists a fqo quasi-Q-nbd of U of x_α such that $U\le f^{-1}(V)$. Now, we have $A\le 1-f^{-1}(V) \Rightarrow A\le 1-U \Rightarrow A \not\mathrm{q} U$.
This is a contradiction since $x_\alpha\in \mathrm{qcl}(A)$.

(d) $\Rightarrow$ (e): Let B be a fuzzy set of Y. Then by (d) we have
$f(\mathrm{qcl}(f^{-1}(B)))\le \mathrm{qcl}(f(f^{-1}(B)))\le \mathrm{qcl}(B)$ and thus $\mathrm{qcl}(f^{-1}(B))\le f^{-1}(\mathrm{qcl}(B))$.

(e) $\Rightarrow$ (a): Let B be any fqc set of Y. By (e), $\mathrm{qcl}(f^{-1}(B))\le f^{-1}(\mathrm{qcl}(B))=f^{-1}(B)$ and so $f^{-1}(B)$ is fqc in X. Hence by Theorem 2.1 f is fuzzy quasi-continuous.

Theorem 2.3. Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be one to one and onto. Then f is fuzzy quasi-continuous if and only if $f(\mathrm{qint}(A)) \le \mathrm{qint}(f(A))$ for each fuzzy set A of X.

Proof. Let A be any fuzzy set of X. Then clearly $f^{-1}(\mathrm{qint}(f(A)))$ is fqo in X. Since f is one to one, we have
$f^{-1}(\mathrm{qint}(f(A)))\le \mathrm{qint}(f^{-1}(f(A)))=\mathrm{qint}(A)$
and $f(f^{-1}(\mathrm{qint}(f(A))))\le f(\mathrm{qint}(A))$.
Since f is onto, we have
$\mathrm{qint}(f(A))=f(f^{-1}(\mathrm{qint}(f(A))))\le f(\mathrm{qint}(A))$.

Conversely, let B be a fqo set of Y. Then $f(\mathrm{qint}(f^{-1}(B)))\ge \mathrm{qint}(f(f^{-1}(B)))=B$ and thus $f^{-1}(f(\mathrm{qint}(f^{-1}(B))))\ge f^{-1}(B)$. Since f is one to one, $\mathrm{qint}(f^{-1}(B))\ge f^{-1}(B)$. This shows that $f^{-1}(B)$ is fqo in X. Hence f is fuzzy quasi-continuous.

Theorem 2.4. Let (X,τ_1,τ_2) and (Y,δ_1,δ_2) be fbts's. If the graph mapping $g:(X,\tau_1,\tau_2)\to(X\times Y,\delta_1,\delta_2)$ of f, where δ_i is the fuzzy product topology generated by τ_i and σ_i (for $i=1,2$), defined by $g(x)=(x,f(x))$ for each $x\in X$, is fuzzy quasi-continuous, then f is fuzzy quasi-continuous.

Proof. Let V be any fqo set of Y. Then by Lemma 2.4 in [1], we have

$$f^{-1}(V)=1\cap f^{-1}(V)=g^{-1}(1\times V).$$

Since $1\times V$ is δ_i-fo set of $X\times Y$ and g is fuzzy quasi-continuous, $f^{-1}(V)$ is fqo set of X.

Remark 2.5. For a fbts (X,τ_1,τ_2), we have $\mathrm{FQ}T_2\Rightarrow \mathrm{FQ}T_1\Rightarrow \mathrm{FQ}T_0$ [12].

Theorem 2.6. Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be one to one. If f is fuzzy quasi-continuous and (Y,σ_1,σ_2) is $\mathrm{FQ}T_k$, then (X,τ_1,τ_2) is $\mathrm{FQ}T_k$, for $k=0,1,2$.

Proof. We give a proof for $k=1$ only; the other cases, being similar, are left. Let x_α and y_β be two distinct fuzzy points in X.
When $x\ne y$, $f(x)\ne f(y)$. Since (Y,σ_1,σ_2) is $\mathrm{FQ}T_1$, there exist quasi-nbds U and V of $f(x_\alpha)$ and $f(y_\beta)$ respectively such that $f(x_\alpha)\not\mathrm{q} V$ and $f(y_\beta)\not\mathrm{q} U$. Since f is fuzzy quasi-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are quasi-nbds of x_α and y_β respectively such that

$x_\alpha \not q f^{-1}(V)$ and $y_\beta \not q f^{-1}(U)$.
When $x=y$ and $\alpha < \beta$ (say), then $f(x)=f(y)$. Since (Y, σ_1, σ_2) is FQT_1, there exists a quasi-Q-nbd V of $f(y_\beta)$ such that $f(x_\alpha) \not q V$. Then $f^{-1}(V)$ is a quasi-Q-nbd of y_β such that $x_\alpha \not q f^{-1}(V)$. Hence (X, τ_1, τ_2) is FQT_1.

A fuzzy set A of a fbts (X, τ_1, τ_2) which can not be expressed as the union of two fuzzy quasi-separated sets is said to be a fuzzy quasi-connected set [12].

Theorem 2.7. Let $f:(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a fuzzy quasi-continuous onto mapping. If A is fuzzy quasi-connected in (X, τ_1, τ_2), then $f(A)$ is fuzzy quasi-connected in (Y, σ_1, σ_2).

Proof. Suppose that $f(A)$ is not fuzzy quasi-connected in (Y, σ_1, σ_2). Then there exist fuzzy quasi-separated sets B and C in Y such that $f(A)=B \cup C$. There exist fqo subsets U and V such that $B \le U$, $C \le V$, $B \not q V$ and $C \not q U$. Since f is fuzzy quasi- continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are fqo in X and
$A=f^{-1}(f(A))=f^{-1}(B \cup C)=f^{-1}(B) \cup f^{-1}(C)$.
Also it can be easily seen that $f^{-1}(B)$ and $f^{-1}(C)$ are fuzzy quasi-separated in X. Thus we arrive at a contradiction.

A mapping $f:(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be fuzzy quasi-open [12] (briefly, fq open) (resp. fuzzy quasi-closed [12] (briefly, fq closed)) if $f(U)$ is fqo (resp. fqc) in Y for each τ_i-fuzzy open (resp. τ_i-fuzzy closed) set U of X.

Theorem 2.8. For a mapping $f:(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following are equivalent:
(a) f is fq open.
(b) $f(A)$ is fqo in Y for each fqo set A of X.
(c) $f(\text{qint}(A)) \le \text{qint}(f(A))$ for each fuzzy set A of X.
(d) $\text{qint}(f^{-1}(B)) \le f^{-1}(\text{qint}(B))$ for any fuzzy set B of Y.

Proof. (a)$\Rightarrow$(b): Let A be any fqo set of X. Then there exist $U \in \tau_1$ and $V \in \tau_2$ such that $A=U \cup V$. Since f is fq open,
$f(A)=f(U \cup V)=f(U) \cup f(V)$ is fqo in Y.
(b)$\Rightarrow$(a): Straightforward.
(b)$\Rightarrow$(c): Let A be any fuzzy set of X. Then by (b) $f(\text{qint}(A))$ is fqo in Y and hence $f(\text{qint}(A)) \le \text{qint}(f(A))$.
(c)$\Rightarrow$(b): Let A be any fqo set of X. Then $f(A)=f(\text{qint}(A)) \le \text{qint}(f(A)) \le f(A)$ and so $f(A)$ is fqo in Y.
(c)$\Leftrightarrow$(d): Straightforward.

Theorem 2.9. If $f:(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is fq open, then for each fuzzy set B of Y and each fqc set A of X such that $f^{-1}(B) \le A$, there is a fqc set C of Y such that $B \le C$ and $f^{-1}(C) \le A$.

Proof. Let B be a fuzzy set of Y and A be a fqc set of X such that $f^{-1}(B) \le A$. Since f is fq open and $1-A$ is fqo in X, $f(1-A)$ is fqo in Y and thus $f(1-A) \le \text{qint}(1-B)=1-\text{qcl}(B)$, i.e. $f^{-1}(\text{qcl}(B)) \le A$. Put $C=\text{qcl}(B)$. Then C is a fqc set of Y such that $B \le C$ and $f^{-1}(C) \le A$.

Theorem 2.10. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be fbts's and $f:(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be one to one and onto. Then f is fq closed if and only if $f^{-1}(\text{qcl}(B)) \le \text{qcl}(f^{-1}(B))$ for each fuzzy set B in Y.

Proof. Straightforward.

Theorem 2.11. Let $f:(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g:(Y, \sigma_1, \sigma_2) \to (Z, \delta_1, \delta_2)$ be mappings.
(a) If f and g are fuzzy quasi-continuous, then $g \circ f$ is fuzzy quasi-continuous.
(b) If $g \circ f$ is fq open and f is fuzzy quasi-continuous and onto, then g is fq open.
(c) If $g \circ f$ is fq open and g is fuzzy quasi-continuous and one to one, then f is fq open.

Proof. Straightforward.

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