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퍼지 몫공간에 관하여

On Fuzzy Qoutient Spaces

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Abstract

In this paper, we introduce the concept of fuzzy quotient spaces as the new ways and investigate their some properties.

Key words ; and phrases : fuzzy quotient space, fuzzy quotient mapping.

1. Introduction and Preliminaries

C.K.Wong, Pu and Liu introduced the concept of fuzzy quotient spaces, respectively and investigated its some properties. In this paper, we introduce the concept of fuzzy quotient spaces as the new ways and study their some properties.

Now we will list some concepts and results with respect to fuzzy set theory and fuzzy topology needed in the next sections.

Let $I=[0,1]$. For a set X, let I^X be the collection of all the mappings from X into I. Then each member of I^X, $A : X \rightarrow I$, is called a fuzzy set in X. In particular, ϕ and X can be considered as fuzzy sets in X defined by $\phi(x)=0$ and $X(x)=1$ for each $x \in X$, respectively. Furthermore, (I^X, $\cup$, $\cap$, c) is a completely distributive lattice for which De Morgan's laws hold(cf.[1,5,12]).

The concept of a fuzzy point in a set its related notions and their properties refer to [5,7,9,11]. We will denote the set of all fuzzy points in a set X as $F_p(X)$.

Definition 1.1[1]. Let $f: X \rightarrow Y$ be a mapping, let $A \in I^X$ and let $B \in I^Y$. Then :

(1) The inverse image of B under f, denoted by $f^{-1}(B)$ is a fuzzy set in X defined by for each $x \in X$, $[f^{-1}(B)](x)=B(f(x))=(B \circ f)(x)$.

(2) The image of A under f, denoted by $f(A)$ is a fuzzy set in Y denoted by for each $y \in Y$,

$$f(A)=\begin{cases} sup_{y=f(x)} A(x), & \text{if } y \in f(X) \\ 0, & \text{if } y \notin f(X) \end{cases}$$

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By the above definition, $f: I^X \rightarrow I^Y$ and $f^{-1} : I^Y \rightarrow I^X$ are mappings.

Result 1.A[1,11]. Let $f : X \rightarrow Y$, let $\{A_\alpha\}_{\alpha \in \Lambda} \subset I^X$ and let $\{B_\alpha\}_{\alpha \in \Lambda} \subset I^Y$. Then :

(1) $f^{-1}(\bigcup_{\alpha \in \Lambda} B_\alpha)=\bigcup_{\alpha \in \Lambda} f^{-1}(B_\alpha)$, $f^{-1}(\bigcap_{\alpha \in \Lambda} B_\alpha)=\bigcap_{\alpha \in \Lambda} f^{-1}(B_\alpha)$.

(2) $f(\bigcup_{\alpha \in \Lambda} A_\alpha)=\bigcup_{\alpha \in \Lambda} f(A_\alpha)$, $f(\bigcap_{\alpha \in \Lambda} A_\alpha)=\bigcap_{\alpha \in \Lambda} f(A_\alpha)$.

(3) $f(f^{-1}(B)) \subset B$, for each $B \in I^Y$.
In particular, if f is surjective, then $f(f^{-1}(B))=B$.

(4) $A \subset f^{-1}(f(A))$, for each $A \in I^X$.
In particular, if f is injective, then $f^{-1}(f(A))=A$.

(5) Let $g: Y \rightarrow Z$ be a mapping. If $B \in I^Z$, then $(g \circ f)^{-1}(B)=f^{-1}(g^{-1}(B))$;
If $A \in I^X$, then $(g \circ f)(A)=g(f(A))$.

(6) If f is bijective, then $[f(A)]^c=f(A^c)$ for each $A \in I^X$.

Definition 1.2[1]. A family $\mathfrak{T}$ of fuzzy sets in a set X is called a fuzzy topology on X if it satisfies the following conditions:

(a) $\phi, X \in \mathfrak{T}$.
(b) If A, $B \in \mathfrak{T}$, then $A \cap B \in \mathfrak{T}$.
(c) If $\{A_\alpha\}_{\alpha \in \Lambda} \subset \mathfrak{T}$, then $\bigcup_{\alpha \in \Lambda} A_\alpha \in \mathfrak{T}$.

The pair $(X, \mathfrak{T})$ is called a fuzzy topological space(in short, fts). Every member of $\mathfrak{T}$ is called a $\mathfrak{T}$-open fuzzy set(in short, F-open set) in X. A fuzzy set A is a $\mathfrak{T}$-closed fuzzy set(in short, F-closed set) if and only if A^c is F-open in X.

Notation 1.3. (1) For a fts X, $FO(X)$ and $FC(X)$ denote the family of all F-open sets and F-closed sets in X, respectively.

(2) For a set X, $\mathfrak{T}_F(X)$ denotes the family of all fuzzy topology on X.

It is clear that $(\mathfrak{T}_F(X), \subset)$ is a complete lattice.

Definition 1.4[5]. Let X be an fts, let $A \in I^X$ and let $x_\lambda \in F_p(X)$. Then A is called :

(1) a fuzzy neighborhood(in short, F-nbd) of x_λ if there exists a F-open set U such that $x_\lambda \in U \subset A$. The family of all the F-nbds of x_λ is called the system of F-nbds of x_λ, and will be denoted by $N_F(x_\lambda)$.

(2) a Q-neighborhood(in short, Q-nbd) of x_λ if there exists a F-open set U such that $x_\lambda q U \subset A$. The family of all the Q-nbds of x_λ is called the system of Q-nbds of x_λ, and will be denoted by $N_Q(x_\lambda)$.

Definition 1.5[4,5]. Let X be a fts and let $A \in I^X$. Then the closure of A, denoted by cl A, is defined by:

$cl\,A = \bigcap\{F \in FC(X) : A \subset F\}$.

It is clear that cl A is the smallest F-closed set containing A and cl(cl A) = $cl\,A$.

Definition 1.6[4,5]. Let X be a fts and let $A \in I^X$. Then the interior of A, denoted by int A, is defined by:

$int\,A = \bigcup\{U \in FO(X) : U \subset A\}$.

It is clear that int A is the largest F-open set contained in A and int(int A) = int A.

Definition 1.7[1]. A mapping $f : (X, \mathfrak{T}) \to (Y, \mathfrak{T}^*)$ is said to be fuzzy continuous(in short, F-continuous) if $f^{-1}(B) \in \mathfrak{T}$ for each $B \in \mathfrak{T}^*$. The mapping f is called a fuzzy homeomorphism(in short, F-homeomorphism) if f is bijective, and both f and f^{-1} are F-continuous.

Result 1.B[1]. If $f : X \to Y$ and $g : Y \to Z$ are F-continuous, then $g \circ f : X \to Z$ is F-continuous.

Definition 1.8[1,8]. Let X and Y be fts's. Then a mapping $f : X \to Y$ said to be :

(1) fuzzy open(in short, F-open) if for each $U \in FO(X)$, $f(U) \in FO(Y)$.

(2) fuzzy closed(in short, F-closed) if for each $F \in FC(X)$, $f(F) \in FC(Y)$.

Definition 1.9[6,8]. Let $(X, \mathfrak{T}_X)$ be fts, let R an equivalence relation on X, let X/R the usual quotient set and let $\pi : X \to X/R$ the usual projection(quotient mapping). Then the collection $\mathfrak{T}_{X/R} = \{B \in I^{X/R} : \pi^{-1}(B) \in \mathfrak{T}_X\}$ is a fuzzy topology on X/R. In this case, $\mathfrak{T}_{X/R}$ is called the fuzzy quotient topology on X/R, the pair $(X/R, \mathfrak{T}_{X/R})$ the fuzzy quotient space of $(X, \mathfrak{T})$ and π the fuzzy quotient mapping.

Result 1.C[8, Theorem 4.1]. (1) $\mathfrak{T}_{X/R}$ is the largest fuzzy topology on X/R for which π is F-continuous.

(2) Let $(Y, \mathfrak{T}_Y)$ be a fts and let $g : (X/R, \mathfrak{T}_{X/R}) \to (Y, \mathfrak{T}_Y)$ a mapping. Then g is F-continuous if and only if $g \circ \pi$ is F-continuous.

2. Fuzzy quotient spaces defined by the first way

After we consider the following result, we introduce the concept of a fuzzy quotient space defined by the first way and study its some properties.

Result 2.A[2]. Let $(X, \mathfrak{T}_X)$ be a fts, let Y a set and let $f: X \to Y$ a mapping. Let $\mathfrak{T}_Y = \{U \in I^Y : f^{-1}(U) \in \mathfrak{T}_X\}$. Then we have the following properties :

(a) $\mathfrak{T}_Y$ is a fuzzy topology on Y.

(b) $f: X \to Y$ is F-continuous.

(c) If U is a fuzzy topology on Y such that $f: X \to (Y, U)$ is F-continuous, then $\mathfrak{T}_Y$ is finer than U.

Definition 2.1. Let $(X, \mathfrak{T}_X)$ be a fts, let Y a set, and let $f: X \to Y$ a surjection. Then $\mathfrak{T}_Y = \{U \in I^Y : f^{-1}(U) \in \mathfrak{T}\}$ is called the fuzzy quotient topology on Y induced by f. The pair $(Y, \mathfrak{T}_Y)$ is called a fuzzy quotient space of X and f a fuzzy quotient mapping.

By Result 2.A, the fuzzy quotient mapping f is not only F-continuous, but $\mathfrak{T}_Y$ is the finest fuzzy topology on Y for which f is F-continuous.

The following result is an immediate consequence of Definition 2.1 :

Proposition 2.2. Let $(Y, \mathfrak{T}_Y)$ be a fuzzy quotient space of a fts $(X, \mathfrak{T}_X)$ with fuzzy quotient mapping f. Then $F \in FC(Y)$ if and only if $f^{-1}(F) \in FC(X)$.

Theorem 2.3. Let $(X, \mathfrak{T}_X)$ and $(Y, \mathfrak{T})$ be fts's, let $f: X \to Y$ F-continuous and surjective and let $\mathfrak{T}_Y$ the fuzzy quotient topology on Y induced by f. If f is F-open or F-closed, then $\mathfrak{T}_Y = \mathfrak{T}$.

(Proof) Suppose f is F-open. By Result 2.A(c), $\mathfrak{T} \subset \mathfrak{T}_Y$. Let $U \in \mathfrak{T}_Y$. Then $f^{-1}(U) \in \mathfrak{T}_X$, by the definition of $\mathfrak{T}_Y$. Since f is F-open, $f(f^{-1}(U)) \in \mathfrak{T}$. Since f is surjective, $f(f^{-1}(U)) = U$. Thus $U \in \mathfrak{T}$. So $\mathfrak{T}_Y \subset \mathfrak{T}$. Hence $\mathfrak{T}_Y = \mathfrak{T}$.

Now suppose f is F-closed. It is sufficent to show that $\mathfrak{T}_Y \subset \mathfrak{T}$. Let $U \in \mathfrak{T}_Y$. Then $f^{-1}(U) \in \mathfrak{T}_X$ by the Definition 2.1 of $\Im_Y$. Thus $[f^{-1}(U)]^c = f^{-1}(U^c) \in FC(X)$. Since f is F-closed, $f(f^{-1}(U^c))$ is F-closed in $(Y, \Im_Y)$. Since f is surjective, $f(f^{-1}(U^c)) = U^c$. Thus U^c is F-closed in $(Y, \Im_Y)$. So $U \in \Im$ and thus $\Im_Y \subset \Im$. Hence $\Im_Y = \Im$.

Theorem 2.3 tells us that if $f : (X, \Im_X) \to (Y, \Im)$ is F-open(or F-closed), F-continuous and surjective, then f is a fuzzy quotient mapping.

Theorem 2.4. The composition of two fuzzy quotient mappings is a fuzzy quotient mapping.

(Proof) Let $f : (X, \Im_X) \to (Y, \Im_Y)$ and $g : (Y, \Im_Y) \to (Z, \Im_Z)$ be fuzzy quotient mappings. Then clearly, by Definition 2.1, $g \circ f : X \to Z$ is surjective. Let $\Im$ be the fuzzy quotient topology on Z induced by $g \circ f$. By Result 2.A(c), $\Im_Z \subset \Im$. Let $W \in \Im$. Then, by Definition 2.1, $(g \circ f)^{-1}(W) \in \Im_X$. Since $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$, $f^{-1}(g^{-1}(W)) \in \Im_X$. Since $\Im_Y$ is the fuzzy quotient topology on Y induced by f, $g^{-1}(W) \in \Im_Y$. Since $\Im_Z$ is the fuzzy guotient topology on Z induced by g, $W \in \Im_Z$. So $\Im \subset \Im_Z$ and thus $\Im = \Im_Z$. Hence $g \circ f$ is a fuzzy quotient mapping.

Theorem 2.5. Let $(X, \Im_X)$ be a fts, let Y a set, let $f: X \to Y$ surjective, let $\Im_Y$ the fuzzy quotient topology on Y induced by f and let $(Z, \Im_Z)$ a fts. Then a mapping $g: Y \to Z$ is F-continuous if and only if $g \circ f: X \to Z$ is F-continuous.

(Proof)($\Rightarrow$): Suppose g is F-continuous. Since $\Im_Y$ is the fuzzy quotiount topology on Y induced by f, f is F-continuous. Hence $g \circ f$ is F-continuous.
($\Leftarrow$): Suppose $g \circ f$ is F-continuous. Let $W \in \Im_Z$. Then $(g \circ f)^{-1}(W) \in \Im_X$ and $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$. Since $\Im_Y$ is a fuzzy quotient topology, $g^{-1}(W) \in \Im_Y$. Hence g is F-continuous.

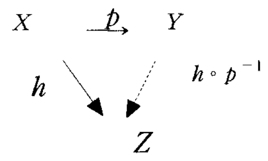
Theorem 2.6. Let $(X, \Im_X)$ and $(Y, \Im_Y)$ be fts's and let $p: X \to Y$ F-continuous and surjective. Then p is a fuzzy quotient mapping if and only if for each fts $(Z, \Im_Z)$ and each mapping $g: Y \to Z$, the F-continuity of $g \circ p$ implies that of g.

(Proof)($\Rightarrow$): Suppose p is a fuzzy quotient mapping. Let $(Z, \Im_Z)$ be a fts, let $g: Y \to Z$ a mapping and let $g \circ p: X \to Z$ F-continuous. Then, by Theorem 2.5, g is F-continuous.
($\Leftarrow$): Suppose the necessary condition holds. Let $\Im$ be the fuzzy quotient topology on Y induced by p. Let p' denote p considered as a mapping from $(X, \Im_X)$ into $(Y, \Im)$ and let $id: (Y, \Im_Y) \to (Y, \Im)$ the identity mapping. Then clearly $id \circ p = p'$ is F-continuous. Thus, by Theorem 2.5, id is F-continuous. On the other hand, $id^{-1} \circ p' = p$ is F-continuous and p' is a fuzzy quotient mapping. Thus id^{-1} is F-continuous. So id is a fuzzy homeomorphism. Hence p is a fuzzy quotient mapping.

Theorem 2.7. Let $(X, \Im_X)$, $(Y, \Im_Y)$ and $(Z, \Im_Z)$ be fts's, let $p: X \to Y$ a fuzzy quotient mapping and let $h: X \to Z$ F-continuous. Suppose $h \circ p^{-1}$ is single-valued ; i.e., for each $y \in Y$, h is constant on $p^{-1}(y)$. Then :

(a) $(h \circ p^{-1}) \circ p = h$ and $h \circ p^{-1}$ is F-continuous.
(b) $h \circ p^{-1}$ is an F-open(F-closed) mapping if and only if $h(U)$ is F-open(F-closed) in Z whenever U is F-open(F-closed) in X satisfying $U = p^{-1}(p(U))$.

(Proof) (a)

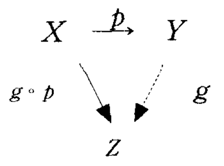


Let $x \in X$. Then $x \in p^{-1}(p(x))$. Since h is constant on $p^{-1}(p(x))$, $h(x) = h(p^{-1}(p(x)))$. But $h(p^{-1}(p(x))) = [(h \circ p^{-1}) \circ p](x)$. So $h = (h \circ p^{-1}) \circ p$. Since h is F-continuous, the F-continuity of $h \circ p^{-1}$ follows from Theorem 2.6.

(b) We will prove the F-open part. The proof of F-closed part is similar to that of F-open part.
($\Rightarrow$): Suppose $h \circ p^{-1}$ is F-open and let U be an F-open set in X such that $U = p^{-1}(p(U))$. Since p is a fuzzy quotient mapping and $p^{-1}(p(U))$ is F-open in X, $p(U)$ is F-open in Y. Since $h \circ p^{-1}$ is F-open, $(h \circ p^{-1})(p(U))$ is F-open in Z. But $(h \circ p^{-1})(p(U)) = h(p^{-1}(p(U))) = h(U)$. Hence $h(U)$ is F-open in Z.
($\Leftarrow$): Suppose the necessary condition holds. Let V be an F-open set in Y. Since p is a fuzzy quotient mapping, $p^{-1}(V)$ is F-open in X. Let $U = p^{-1}(V)$. Then $V = p(U)$. So $U = p^{-1}(V) = p^{-1}(p(U))$. Thus, by the hypothesis, $h(U)$ is F-open in Z. On the other hand, $(h \circ p^{-1})(V) = h(p^{-1}(p(U))) = h(U)$. Thus $(h \circ p^{-1})(V)$ is F-open in Z. Hence $h \circ p^{-1}$ is F-open.

Theorem 2.8. Let $(X, \Im_X)$, $(Y, \Im_Y)$ and $(Z, \Im_Z)$ be fts, let $p: X \to Y$ a fuzzy quotient mapping and let $g: Y \to Z$ a surjection. Then $g \circ p$ is a fuzzy quotient mapping if and only if g is a fuzzy quotient mapping.

(Proof)



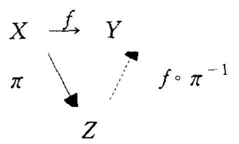
By Theorem 2.6, $g \circ p$ is F-continuous if and only if g is F-continuous. Therefore, for each $W \in \mathfrak{I}_Z$, $(g \circ p)^{-1}(W)$ is F-open in X if and only if $g^{-1}(W)$ is F-open in Y.

Let $f: X \rightarrow Y$ be a mapping. Define a relation $\sim_f$ and X by $x \sim_f y$ if and only if $f(x) = f(y)$. Then $\sim_f$ is an equivalence relation on X. In this case, $\sim_f$ is called the equivalence relation on X induced by f.

Lemma 2.9. Let $f: X \rightarrow Y$ be a F-continuous mapping, let $\sim_f$ the equivalence relation on X induced by f and let $\pi: X \rightarrow X/\sim_f$ the natural mapping. Let $(X/\sim_f, \mathfrak{I}_{X/\sim_f})$ be a fuzzy quotient space of X. Then $f \circ \pi^{-1}$ is F-continuous and injective.

Furthermore, if f is surjective, then $f \circ \pi^{-1}$ is bijective.

(Proof)



Clearly $\pi: X \rightarrow X/_{\sim_f}$ is a fuzzy quotient mapping. Let $y \in Y$. Then $f^{-1}(y) = \emptyset$ or $f^{-1}(y) \neq \emptyset$. If $f^{-1}(y) = \emptyset$, then clearly π is constant on $f^{-1}(y)$. Suppose $f^{-1}(y) \neq \emptyset$ and let $x_1, x_2 \in f^{-1}(y)$. Then $f(x_1) = y = f(x_2)$. Thus $x_1 \sim_f x_2$. So $[x_1] = [x_2]$ and thus $\pi(x_1) = \pi(x_2)$. Hence π is constant on $f^{-1}(y)$. i.e., $f \circ \pi^{-1}$ is single-valued. Therefore, by Theorem 2.7(a), $f \circ \pi^{-1}$ is F-continuous.

Now suppose $[x_1], [x_2] \in X/_{\sim_f}$ and $(f \circ \pi^{-1})([x_1]) = (f \circ \pi^{-1})([x_2])$. Then $f(\pi^{-1}([x_1])) = f(\pi^{-1}([\pi_2]))$. Let $y_1 \in \pi^{-1}([x_1])$ and $y_2 \in \pi^{-1}([x_2])$. Then $f(y_1) = f(y_2)$. Thus $y_1 \sim_f y_2$. So $[x_1] = \pi(y_1) = \pi(y_2) = [x_2]$. Hence $f \circ \pi^{-1}$ is injective.

Finally suppose f is surjective and let $y \in Y$. Then there exists $x \in X$ such that $f(x) = y$. Thus $[x] \in X/\sim_f$ and $(f \circ \pi^{-1})([x]) = y$. So $f \circ \pi^{-1}$ is surjective. Hence $f \circ \pi^{-1}$ is bijective.

Theorem 2.10. Let $(X, \mathfrak{I}_X)$ and $(Y, \mathfrak{I}_Y)$ be fts's and let $f: X \rightarrow Y$ be F-continuous and surjective. Then $f \circ \pi^{-1}: X/\sim_f \rightarrow Y$ is an F-homeomorphism if and only if f is a fuzzy quotient mapping.

(Proof)($\Leftarrow$): Suppose f is a fuzzy quotient mapping. Then, by the lemma 2.9, $f \circ \pi^{-1}$ is F-continuous and bijective. It is sufficient to show that $f \circ \pi^{-1}$ is F-open. Let U be an F-open set in X such that $U = \pi^{-1}(\pi(U))$. Then $\pi^{-1}(\pi(U)) = f^{-1}(f(U))$. Thus $f^{-1}(f(U))$ is F-open in X. Since f is a fuzzy quotient mapping, $f(U)$ is F-open in Y. So, by Theorem 2.7 (b), $f \circ \pi^{-1}$ is F-open. Hence $f \circ \pi^{-1}$ is an F-homeomorphism.

($\Rightarrow$): Suppose $f \circ \pi^{-1}$ is an F-homeomorphism. Then $f \circ \pi^{-1}$ is fuzzy quotient mapping. By Theorem 2.8, $(f \circ \pi^{-1}) \circ \pi$ is a fuzzy quotient mapping. But $(f \circ \pi^{-1}) \circ \pi = f$. Hence f is a fuzzy quotient mapping.

3. Fuzzy quotient spaces defined by the second way

We turn our attention toward the second way of defining a fuzzy quotient space.

Definition 3.1. Let $(X, \mathfrak{I}_X)$ be a fts and let D a partition of X. Define a mapping $p: X \rightarrow \mathrm{D}$ as follows : For each $x \in X$, $p(x) \in \mathrm{D}$ contains x. Let $\mathfrak{I}\mathrm{D}$ be the fuzzy quotient topology on D induced by p. Then $(\mathrm{D}, \mathfrak{I}\mathrm{D})$ is called a fuzzy quotient space of X. The mapping p is called the natural mapping of X onto D. The set D is also called a decomposition of X and the fuzzy quotient space $(\mathrm{D}, \mathfrak{I}\mathrm{D})$ is also called a fuzzy decomposition space or a fuzzy identification space of X.

Example 3.2. Let $X = \{a, b, c\}$, let $\mathrm{D} = \{\{a, b\}, \{c\}\}$, and let $\mathfrak{I}_X = \{\emptyset, X, O_1, O_2, O_1 \cap O_2, O_1 \cup O_2\}$, where $O_1 = \{(a, 0.3), (b, 0.3), (c, 0.8)\}$ and $O_2 = \{(a, 0.6), (b, 0.6), (c, 0.7)\}$. Let the natural mapping $p: X \rightarrow \mathrm{D}$ be defined by :

$p(a) = p(b) = \{a, b\}$ and $p(c) = \{c\}$.

Then the fuzzy quotient topology $\mathfrak{I}\mathrm{D}$ on D induced by p is as follows :

$\mathfrak{I}\mathrm{D} = \{\emptyset, X, O_1^*, O_2^*, O_1^* \cap O_2^*, O_1^* \cup O_2^*\}$, where $O_1^* = \{(\{a, b\}, 0.3), (\{c\}, 0.8)\}$ and $O_2^* = \{(\{a, b\}, 0.6), (\{c\}, 0.7)\}$.

Lemma 3.3. let X be a set, let D a partition of X and let $p: X \rightarrow \mathrm{D}$ the natural mapping. Then for each $\xi \in I^{\mathrm{D}}$, $p^{-1}(\xi) = \bigcup \xi$.

(Proof) Let $x_\lambda \in \bigcup \xi$. Then

$$\lambda \leq (\bigcup \xi)(x) = \sup_{x \in D \in \mathscr{D}} \xi(D) = \sup_{x \in p(x) \in \mathscr{D}} \xi(p(x))$$

Thus $\lambda \leq \xi(p(x)) = p^{-1}(\xi)(x)$. So $x_\lambda \in p^{-1}(\xi)$ and thus $\bigcup\xi \subset p^{-1}(\xi)$. Now let $x_\lambda \in p^{-1}(\xi)$. Then $\lambda \le p^{-1}(\xi)(x) = \xi(p(x))$. Since p is the natural mapping, $x \in p(x) \in \mathrm{D}$.

Thus $\lambda \le sup\, x \in p(x) \in \mathrm{D}\ \xi(p(x)) = (\bigcup\xi)(x)$.

So $x_\lambda \in \bigcup\xi$ and thus $p^{-1}(\xi) \subset \bigcup\xi$. Hence $p^{-1}(\xi) = \bigcup\xi$.

Proposition 3.4. Let $(X, \Im_X)$ be a fts, let D a decomposition of X, let $p: X \to \mathrm{D}$ the natural mapping, let $\Im\mathrm{D}$ the fuzzy quotient topology on D and let $\xi \in I^{\mathrm{D}}$. Then $\xi \in \Im\mathrm{D}$ if and only if $\bigcup\xi \in \Im_X$.

(Proof) It is clear from Lemma 3.3.

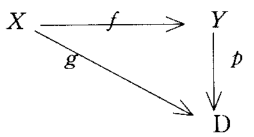
Theorem 3.5. let $(X, \Im_X)$ be a fts, let Y a set, let $f: X \to Y$ a surjection, and let $\Im_Y$ the fuzzy qoutient topology on Y induced by f. Let $\mathrm{D} = \{f^{-1}(y) : y \in Y\}$. Then there exists a F-homeomorhism $g: Y \to \mathrm{D}$ such that $g \circ f = p$.

(Proof) Clearly D is a partition of X. Define $g: Y \to \mathrm{D}$ by $g(y) = f^{-1}(y)$ for each $y \in Y$. Then clealy g is bijective. Let $x \in X$. Then $g(f(x)) = f^{-1}(f(x))$. Thus $x \in g(f(x)) \in \mathrm{D}$. So $g(f(x)) = p(x)$. Hence $g \circ f = p$.

Let $\xi \in \Im\mathrm{D}$. Since p is F-continuous and $g \circ f = p$, $(g \circ f)^{-1}(\xi) = f^{-1}(g^{-1}(\xi)) \in \Im_X$. Since $\Im_Y$ is the fuzzy quotient topology on Y induced by f, $g^{-1}(\xi) \in \Im_Y$. So g is F-continuous.

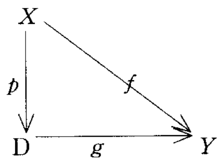
Now let $U \in \Im_Y$. Then, by definition 2.1, $f^{-1}(U) \in \Im_X$. On the other hand, $g(U) = f^{-1}(U)$. Thus $g(U) \in \Im_X$. So g is F-open. Therefore g is a F-homeomorhism.

Theorem 3.5 yields the following commutative diagram, where g is a F-homeomorphism :



The following theorem provides a criterion for determining when a mapping whose domain is a fuzzy quotient space is F-continuous.

Theorem 3.6. Let $(X, \Im_X)$ be a fts, let $(\mathrm{D}, \Im\mathrm{D})$ a fuzzy decomposition space of X and let $p: X \to \mathrm{D}$ the natural mapping. Let $(Y, \Im_Y)$ be a fts and let $f: X \to Y$ a F-continuous mapping such that for each $D \in \mathrm{D}$, $f(x) = f(y)$ for all $x, y \in D$. Then there exists a F-continuous mapping $g: \mathrm{D} \to Y$ such that $g \circ p = f$.



(Proof) Define g: $\mathrm{D} \to Y$ by $g(D) = f(D)$ for each $D \in \mathrm{D}$. Let $x \in X$. Then $g(p(x)) = f(x)$, since $x \in p(x) \in \mathrm{D}$. So $g \circ p = f$. Let $V \in \Im_Y$. Then $f^{-1}(V) = (g \circ p)^{-1}(V) = p^{-1}(g^{-1}(V)) \in \Im_X$. Since p is the fuzzy quotient mapping, $g^{-1}(V) \in \Im\mathrm{D}$. Hence g is F-continuous such that $g \circ p = f$.

Theorem 3.7. Let $(X, \Im_X)$ and $(Y, \Im_Y)$ be fts's, let $f: X \to Y$ F-continuous surjective, let $\mathrm{D} = \{f^{-1}(y) : y \in Y\}$, and let $\Im\mathrm{D}$ the fuzzy quotient topology on D induced by the natural mapping $p: X \to \mathrm{D}$. Then the induced F-continuous mapping(given in Theorem 3.6) $g: \mathrm{D} \to Y$ is a F-homeomorphism if and only if f is a fuzzy quotient mapping.

(Proof)($\Rightarrow$): Suppose g is a F-homeomorphism. Then clearly g and p are fuzzy quotient mappings. Thus, by Theorem 2.4, $g \circ p$ is a fuzzy quotient mapping. But $g \circ p = f$. Hence f is a fuzzy quotient mapping.

($\Leftarrow$): Suppose f is a fuzzy qoutient mapping. By Theorem 3.6, g is F-continuous. Observe that g is bijective. let $V \in \Im\mathrm{D}$. Then $p^{-1}(V) \in \Im_X$ and $f^{-1}(g(V)) = p^{-1}(V)$. Thus $f^{-1}(g(V)) \in \Im_X$. Since f is a fuzzy quotient mapping, $g(V) \in \Im_Y$. So g is F-open. Hence g is a F-homeomorphism.

The following theorem gives a criterion for a fuzzy quotient space to be a fuzzy Hausdorff space in the sense of Pu and Liu.

Theorem 3.8. Let $(X, \Im_X)$ and $(Y, \Im_Y)$be fts, let $f: X \to Y$ F-continuous surjective, let $\mathrm{D} = \{f^{-1}(y) : y \in Y\}$, and let $\Im\mathrm{D}$ the fuzzy quotient topology on D induced by the natural mapping $p: X \to \mathrm{D}$. If Y is FT_2 in the sense of Pu and Liu, then so is D.

(Proof) By Theorem 3.6, there exists a F-continuous mapping $g: \mathrm{D} \to Y$. Moreover g is bijective. Let D_λ and E_μ be distincte fuzzy points in D. Then $g(D_\lambda)$ and $g(E_\mu)$ are distinct fuzzy points in Y. Since Y is FT_2, there exist $U, V \in \Im_Y$ such that $g(D_\lambda) q U$, $g(E_\mu) q V$ and $U \cap V = \emptyset$. Since g is F-continuous, $g^{-1}(U)$, $g^{-1}(V) \in \Im\mathrm{D}$. Moreover $g^{-1}(U) \cap g^{-1}(V) = \emptyset$, $D_\lambda q g^{-1}(U)$ and $E_\mu q\, g^{-1}(V)$. Hence D is FT_2.

Remark 3.9. Definition 1.9 is the third way of defining a fuzzy quotient space.

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