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A Fuzzy System Representation of Functions of Two Variables and its Application to Gray Scale Images

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Abstract

An approximate representation of discrete functions $\{f_{i,j}|i,j=-1,0,1,\cdots,N+1\}$in two variables by a fuzzy system is described. We use the cubic B-splines as fuzzy sets for the input fuzzification and spike functions as the output fuzzy sets. The ordinal number of $f_{i,j}$ in the sorted list is taken to be the out put fuzzy set number in the (i,j)th entry of the fuzzy rule table. We show that the fuzzy system is an exact representation of the cubic spline function $S(x,y)=\sum_{i,j=-1}^{N+1} f_{i,j}B_i(x)B_j(y)$ and that the approximation error $S(x,y)-f(x,y)$ is surprisingly $O(h^2)$ when $f(x,y)$ is three times continuously differentiable. We prove that when $f(x,y)$ is a gray scale image, then the fuzzy system is a smoothed representation of the image and the original image can be recovered exactly from its fuzzy system representation when it is a digitized image.

Key words : Fuzzy System ; Function Representation ; Spline Interpolation; B-splines; Digital Images ; Discrete Functions

1. Introduction

We have shown in our earlier work[1] that there is a practical and easy method to represent a cubic spline interpolation function by a fuzzy system. Considering the fact that an arbitrary continuously differentiable function can be approximated by a cubic spline interpolation function on a bounded interval within a prescribed accuracy, it is clear that any continuously differentiable function can be represented by a fuzzy system.

It is known also that fuzzy systems can be used to approximate continuous functions on a compact set with in arbitrary accuracy. Kosko[2] and Castro et. al[3] showed that for a continuous function $f(x)$ on a compact set and an $\varepsilon>0$, there exists a fuzzy system that approximates $f(x)$ within ε. L. Wang[4] also shows this by proving that a set of fuzzy systems, each of which being identified as a function of fixed form with different parameter values, is dense in the set of all continuous functions.

In the following, we will consider functions of two variables $f(x,y)$ on an interval $[a,b]\times[c,d]$.
For convenience, we assume $d-c=b-a$ and divide the intervals $[a,b]$ and $[c,d]$ into n subintervals. Let $x_j=a+jh$, $y_j=c+jh$ with $h=\frac{b-a}{n}$ and let $B_i(t)$'s be the cubic B-spline functions defined on $[t_{i-2},t_{i+2}]$ by

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$$\frac{1}{6h^3}\times\begin{cases}(t-t_{i-2})^3,\\ h^3+3h^2(t-t_{i-1})+3h(t-t_{i-1})^2-3(t-t_{i-1})^3,\\ h^3+3h^2(t_{i+1}-t)+3h(t_{i+1}-t)^2-3(t_{i+1}-t)^3,\\ (t_{i+2}-t)^3,\end{cases}$$

for t in the intervals $[t_{i-2},t_{i-1}]$, $[t_{i-1},t_i]$, $[t_i,t_{i+1}]$, and $t\in[t_{i+1},t_{i+2}]$ respectively with $i=-1,0.1.\cdots,N+1$.

Then the spline interpolation function for $f(x,y)$ can be written as $S(x,y)=\sum_{i,j=-1}^{N+1} c_{i,j}B_i(x)B_j(y)$, with $c_{i,j}$'s obtained from solving a set of linear equations for the interpolation constraints[5]. In our earlier work[1], we showed that if one uses the following procedure to set up a fuzzy system for cubic spline interpolation function of the form $S(x,y)=\sum_{i,j=-1}^{N+1} c_{i,j}B_i(x)B_j(y)$\$, then the fuzzy system evaluates exactly the same values as $S(x,y)$.

1.1 Fuzzy Sets for Input Fuzzification

Let $x_i=a+ih$, $y_j=c+jh$ with $h=\frac{b-a}{n}$ where we assumed $d-c=b-a$. If we define $B_i(x)$ and $B_j(y)$ as in (1), then the B-spline function $B_i(x)$ has support $[x_{i-2},x_{i+2}]$ with the maximum value of $\frac{2}{3}$ at x_i. We take these functions $B_i(x)$ and $B_j(y)$ for $i,j=-1,0,1,\cdots,N+1$ as fuzzy sets for input variables x and y respectively.

1.2 Generation of Fuzzy Rules

Given a cubic spline function $S(x,y)=\sum_{i,j=-1}^{N+1} c_{i,j}B_i(x)B_j(y)$, we sort the $(n+3)^2$ coefficients $c_{i,j}$ in an increasing order and delete the duplicate ones, i.e., delete $c_{k,l}$ for

$(f_{k+1,j}-f_{k-1,j})B_j(y_l)$ which is equal to $\frac{1}{12}(f_{k+1,l-1}-f_{k-1,l-1})+4(f_{k+1,l}-f_{k-1,l})+(f_{k+1,l+1}-f_{k-1,l+1})\}$. Applying Lemma 4 to each of the three terms inside the parentheses, we obtain $S_x(x_k,y_l)=\frac{1}{6}\times(f_x(x_k,y_{l-1})+4f_x(x_k,y_l)+f_x(x_k,y_{l+1}))+O(h^2)$. Now, note that $f_x(x_k,y_{l-1})+f_x(x_k,y_{l+1})=2f_x(x_k,y_l)+O(h^2)$ and hence we have the result. Q.E.D.

Lemma 7. Let $f(x,y)$ and $S(x,y)$ be as defined above. Then we have $S_{xx}(x_k,y_l)-f_{xx}(x_k,y_l)=O(h)$ and $S_{xy}(x_k,y_l)-f_{xy}(x_k,y_l)=O(h)$.

Proof. Substituting $B''_i(t_{i-1})=B''_i(t_{i+1})=\frac{1}{h^2}, B''_i(t_i)=-\frac{2}{h^2}$, we have $S_{xx}(x_k,y_l)=\sum_{i,j=-1}^{N+1} f_{i,j}B''_i(x_k)B_j(y_l)=\frac{1}{h^2}\sum_{j=-1}^{N+1}(f_{k-1,j}-2f_{k,j}+f_{k+1,j})\times B_j(y_l)$, which can be written as $\frac{1}{6h^2}\times(f_{k-1,l-1}-2f_{k,l-1}+f_{k+1,l-1})+4(f_{k-1,l}-2f_{k,l}+f_{k+1,l})+f_{k+1,l}(f_{k-1,l+1}-2f_{k,l+1}+f_{k+1,l+1})\}$. Now, use the relation $f_{k-1,j}-2f_{k,j}+f_{k+1,j}=h^2f_{xx}(x_k,y_l)+O(h^3)$ to obtain $S_{xx}(x_k,y_l)=f_{xx}(x_k,y_l)+O(h)$ and hence the first part is proved. A similar proof for the second part is omitted. Q.E.D.

Lemma 8. Let $f(x,y)$ and $S(x,y)$ be as defined above. Then we have $S_x(x_k,y)-f_x(x_k,y)=O(h^2)$ for all $y\in[c,d]$.

Proof. Without loss of generality, we may assume $y\in[y_l,y_{l+1})$ for some l. Note that $S_x(x_k,y)-f_x(x_k,y)=(S_x(x_k,y)-S_x(x_k,y_l))+(S_x(x_k,y_l)-f_x(x_k,y_l))+(f_x(x_k,y_l)-f_x(x_k,y))$. Using $S_x(x_k,y)-S_x(x_k,y_l)=(y-y_l)S_{xy}(x_k,y_l)+\frac{(y-y_l)^2}{2}S_{xyy}(x_k,y_l)+O(h^3)$ and a similar relation for $f_x(x_k,y)$, we obtain the conclusion. Q.E.D.

Theorem 3. If $f(x,y)$ is a three times continuously differentiable function and if $S(x,y)=\sum_{i,j=-1}^{N+1} f_{i,j}B_i(x)B_j(y)$, then we have $S(x,y)-f(x,y)=O(h^2)$ for all $(x,y)\in[a,b]\times[c,d]$.

Proof. Assume $x\in[x_k,x_{k+1})$ and $y\in[y_l,y_{l+1})$ for some k and l. We write $S(x,y)-f(x,y)$ as a sum of the four terms ; $S(x,y)-S(x_k,y)$, $S(x_k,y)-S(x_k,y_l)$, $S(x_k,y_l)-f(x_k,y_l)$, $f(x_k,y_l)-f(x_k,y)$. When the first and the fifth are added, we obtain $(x-x_k)(S_x(x_k,y)-f_x(x_k,y))+O(h^3)$ which is $O(h^3)$. Similarly, the second and the fourth are added to become $O(h^3)$ and the third is of $O(h^2)$ by Lemma 6. Q.E.D.

4. Application Examples

In this section, we consider the case where $f_{i,j}\in\{0,1\}$ for all $i,j=-1,0,1,\cdots,N+1$, i.e. when $f(x,y)$ is a discrete function for a digitized image with 0's for white pixels and 1's for black pixels. We will show that from the approximate continuous representation of the digitized image, i.e. $S(x,y)=\sum_{i,j=-1}^{N+1} f_{i,j}B_i(x)B_j(y)$, one can recover the original image $\{f_{i,j}, i,j=-1,0,\cdots,N+1\}$.

Lemma 9. Let $S(x,y)=\sum_{i,j=-1}^{N+1} f_{i,j}B_i(x)B_j(y)$, then we have $(S_{xx}+S_{yy})(x_k,y_l)=\frac{1}{3h^2}(f_{k-1,l-1}+f_{k-1,l}+f_{k-1,l+1}+f_{k,l-1}-8f_{k,l}+f_{k,l+1}+f_{k+1,l-1}+f_{k+1,l}+f_{k+1,l+1})$

Proof. This follows from a routine computation using $B_k(x_{k-1})=B_k(x_{k+1})=\frac{1}{6}, B_k(x_k)=\frac{2}{3}, B''_k(x_{k-1})=B''_k(x_{k+1})=\frac{1}{h^2}, B''_k(x_k)=-\frac{2}{h^2}$, while $B_k(x_j)=B''_k(x_j)=0$ for all other indices j. Q.E.D.

Lemma 10. If $f_{i,j}\in\{0,1\}$ for all $i,j=-1,0,1,\cdots,N+1$, then we have $|S(x_k,y_l)-f_{k,l}|\le\frac{4}{9}$.

Proof. From Theorem 3, it is clear that $|S(x_k,y_l)-f_{k,l}|\le\frac{4}{9}+O(h^2)$ for all $h>0$. Hence, we must have $|S(x_k,y_l)-f_{k,l}|\le\frac{4}{9}$. Q.E.D.

The following theorem states that if a digitized image is represented by a fuzzy system $S(x,y)$ using $f_{i,j}$'s as its rule table, then the pixel value at (x_k,y_l) can be recovered by evaluating $[S(x_k,y_l)+0.5]$ where the brackets are for the Gaussian bracket function.

Theorem 4. Let $\{f_{i,j}|i,j=-1,0,\cdots,N+1\}$ be a digitized image and let $S(x,y)=\sum_{i,j=-1}^{N+1} f_{i,j}B_i(x)\ B_j(y)$ be the continuous cubic spline representation of the image. If we define $g_{k,l}=[S(x_k,y_l)+0.5]$, then $g_{k,l}=f_{k,l}$ for all $k,l=-1,0,1,\cdots,N+1$.

Proof. By Lemma 8, we have $-\frac{4}{9}\le S(x_k,y_l)-f_{k,l}\le\frac{4}{9}$. Hence, we have $f_{k,l}+\frac{0.5}{9}\le S(x_k,y_l)+0.5\le f_{k,l}+\frac{8.5}{9}$ from which we obtain $[f_{k,l}+\frac{0.5}{9}]\le[S(x_k,y_l)+0.5]\le[f_{k,l}+\frac{8.5}{9}]$. Now, note that $[f_{k,l}+\frac{0.5}{9}]=[f_{k,l}]$ and $[f_{k,l}+\frac{8.5}{9}]=[f_{k,l}]$ since $f_{k,l}\in\{0,1\}$. Q.E.D.

5. Conclusion

The fuzzy system we designed provides a very efficient method to evaluate cubic spline functions. We proved that when a set of discrete points is given in (x, y) coordinates, then a cubic spline function can be defined to approximate the function with approximation error $O(h^2)$. Even though the approximation error is not as good as the cubic spline interpolation error which is of $O(h^4)$, our representation of the function does not require any computation at all including the matrix inversion which is necessary for spline interpolations. It is expected that our fuzzy system representation for functions given in discrete points can be utilized in various image processing techniques.

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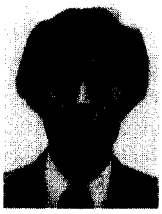

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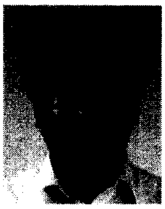

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