Asymptotic Relative Efficiencies of Chaudhuri's Estimators for the Multivariate One Sample Location Problem

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Abstract

We derive the asymptotic relative efficiencies in two special cases of Chaudhuri's estimators for the multivariate one sample problem. And we compare those two when observations are independent and identically distributed from a family of spherically symmetric distributions including normal distributions.

Keywords : Hotelling's T^2, Direction Vectors, Asymptotic Relative Efficiencies.

1. Introduction

In one sample multivariate location problem, the conventional parametric test statistic is Hotelling's T^2 which assumes that observations are independent and identically distributed (i.i.d.) from multivariate normal distributions. However the normality assumption is often violated. So, to avoid distributional assumptions, Raleigh (Watson, 1983), Randles(1989) and Chaudhuri(1992, 1993) adopted direction vectors. Under the simple symmetry assumptions adopting direction vectors makes the test statistics free of distributions.

In this paper, we review some important asymptotic properties of those location estimators. Under the more general spherically symmetric distributions with heavy or light tails, we obtain Chaudhuri estimator's asymptotic relative efficiencies (ARE) with respect to Hotelling's T^2 for two cases : (i) when the direction vector of each observation is used (this corresponds to $L=1$) and (ii) when the direction vector of average of two observations is used (this corresponds to $L=2$).

Chaudhuri (1992) discussed the AREs of his estimators in terms of the inverse of the variances. Even though the inverse of variances gives a correct view of AREs' tendency, they are different from the exact AREs by a constant. Also the variances were obtained only under

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the normality assumption. Thus it would be meaningful to derive exact AREs and calculate them under more general distributional assumptions.

Finally we compare two estimators in terms of efficiencies and evaluate which Chaudhuri's estimator works better. AREs are obtained based on the previous results by Randles(1989) and Choi and Marden (1997). In both of the works, direction vectors were used.

2. Chaudhuri's location estimators for the multivariate one sample

Let $X_1, X_2, \cdots, X_n$ be i.i.d. multivariate observations in R^p for $p>0$ and $\|\cdot\|$ be the notation for the Euclidean norm. Let θ be its location parameter. Without loss of generality, we test the hypothesis $H_o : \theta = 0$ against $H_A : \theta \neq 0$. For $L \leq n$, let $A_n^{(L)}$ be the set of subsets of $\{1, 2, \cdots, n\}$ with size L, that is,

$$A_n^{(L)} = \{\alpha | \alpha \subseteq \{1, 2, \cdots, n\} \text{ and } size(\alpha) = L\}. \tag{2.1}$$

Define $\overline{X}_\alpha = (1/L) \sum_{i \in \alpha} X_i$. To estimate the location parameter of the sample, Chaudhuri (1992, 1993) uses the average direction vectors U_L as follows:

$$U_L = \frac{1}{size(A_n^{(L)})} \sum_{\alpha \in A_n^{(L)}} \frac{\overline{X}_\alpha}{\|\overline{X}_\alpha\|}. \tag{2.2}$$

Using U-statistic theory he found the asymptotic distribution of U_L. Let us define some useful expectations beforehand. They are as follows:

$$D_1^{(L)} \equiv E_{H_o} \left[\frac{1}{\|\sum_{i=1}^{L} X_i/L\|} \left(I_p - \frac{\sum_{i=1}^{L} X_i/L}{\|\sum_{i=1}^{L} X_i/L\|} \frac{\sum_{i=1}^{L} X_i^T/L}{\|\sum_{i=1}^{L} X_i/L\|} \right) \right] \tag{2.3}$$

and

$$D_2^{(L)} \equiv E_{H_o} \left[\frac{X_1 + X_2 + \cdots + X_L}{\|X_1 + X_2 + \cdots + X_L\|} \frac{(X_1 + X_2^* + \cdots + X_L^*)}{\|X_1 + X_2^* + \cdots + X_L^*\|} \right], \tag{2.4}$$

where $X_1, X_2, X_2^*, \cdots, X_L, X_L^*$ are i.i.d. multivariate observations. If a sample is from a spherically symmetric distribution, then $\|x\|$ is independent of $\frac{x}{\|x\|}$ and

$E\left[\frac{x}{\|x\|} \frac{x^T}{\|x\|} \right] = \frac{1}{p} I_p$. Thus $D_1^{(L)} = E_{H_o}\left[1/\|\sum_{i=1}^{L} X_i/L\| \right]\left(1 - \frac{1}{p} \right)I_p$. Based on these, Chaudhuri introduced the multivariate version of Lth order Hodges-Lehmann type (1963) location estimators as follows:

$$\widehat{\theta_n^{(L)}} = \frac{1}{C(n, L)} [D_1^{(L)}]^{-1} \sum_{\alpha \in A_n^{(L)}} \frac{\overline{X}_\alpha}{\|\overline{X}_\alpha\|}, \tag{2.5}$$

where $C(n, L) = \dfrac{n!}{(n-L)! L!}$. Then $\sqrt{n}\ \widehat{\theta_n^{(L)}} \to N(0, \Sigma_{U_L})$ in distribution, where the variance-covariance matrix is

$$\Sigma_{U_L} = L^2 [D_1^{(L)}]^{-1} D_2^{(L)} [D_1^{(L)}]^{-1}. \tag{2.6}$$

Chaudhuri (1992) obtained a closed form of $D_1^{(L)}$ under the spherical normality. He mentioned that for the general $L \geq 2$ and the high dimension, $D_2^{(L)}$ can be obtained from (p-1)-fold Lebesque integrals of the spherically symmetric density function $f(x)$ by fixing one component as 0. $D_2^{(L)}$ was expressed as an integral form especially when the distribution is bivariate normal with a diagonal variance-covariance matrix. Hence assuming the spherical normality, the asymptotic variance-covariance $\sigma^2(p, L)$ of $\sqrt{n}\ \widehat{\theta_n^{(L)}}$ on R^p, for $p \geq 2$ was obtained in a closed form. And it was shown that $\sigma^2(p, L) > \sigma^2(p+1, L)$ and $\sigma^2(p, L) > \sigma^2(p, L+1)$ for fixed L and p respectively. This implies that under the spherically symmetric normal distribution for a fixed L its efficiency increases as the dimension p increases, and for a fixed p it increases as L increases. This implies that under the normality assumption his statistics work better in the higher dimension than in the lower dimension, and the statistic of $(L+1)$ is better than that of L when the dimension is fixed.

Here two location estimators of our interest are $\widehat{\theta_n^{(1)}}$ and $\widehat{\theta_n^{(2)}}$. Let us write them down explicitly.

Case : $L = 1$

$$U_1 = \frac{1}{n} \sum_{i=1}^{n} \frac{X_i}{\|X_i\|}, \tag{2.7}$$

$$D_1^{(1)} = E_{H_o}\left[\frac{1}{\|X_1\|} \left(I_p - \frac{X_1}{\|X_1\|} \frac{X_1^T}{\|X_1\|} \right) \right] \tag{2.8}$$

and

$$D_2^{(1)} = E_{H_o}\left[\frac{X_1}{\|X_1\|} \frac{X_1^T}{\|X_1\|} \right], \tag{2.9}$$

where X_1 is a multivariate random variable from any distribution. The 1st order location estimator is

$$\widehat{\theta_n^{(1)}} = \frac{1}{n} [D_1^{(1)}]^{-1} \sum_{i=1}^{n} \frac{X_i}{\|X_i\|}. \tag{2.10}$$

Then $\sqrt{n}\ \widehat{\theta_n^{(1)}} \to N(0, \Sigma_{U_1})$ in distribution, where $\Sigma_{U_1} = [D_1^{(1)}]^{-1} D_2^{(1)} [D_1^{(1)}]^{-1}$.

<u>Case : $L = 2$</u>

$$U_2 = \frac{2}{n(n-1)} \sum_{i<j} \frac{X_i + X_j}{\|X_i + X_j\|},$$ (2.11)

$$D_1^{(2)} = E_{H_o}\left[\frac{1}{\|(X_1 + X_2)/2\|}\left(I_p - \frac{X_1 + X_2}{\|X_1 + X_2\|}\frac{(X_1 + X_2)^T}{\|X_1 + X_2\|} \right) \right]$$ (2.12)

and

$$D_2^{(2)} = E_{H_o}\left[\frac{X_1 + X_2}{\|X_1 + X_2\|}\frac{(X_1 + X_3)^T}{\|X_1 + X_3\|} \right],$$ (2.13)

where X_1, X_2, X_3 are i.i.d. from a multivariate distribution. The 2nd order location estimator is

$$\widehat{\theta_n^{(2)}} = \frac{2}{n(n-1)}[D_1^{(2)}]^{-1}\sum_{i<j}\frac{X_1 + X_2}{\|X_1 + X_2\|}.$$ (2.14)

Then $\sqrt{n}\ \widehat{\theta_n^{(2)}} \to N(0, \Sigma_{U_2})$ in distribution, where $\Sigma_{U_2} = 2^2[D_1^{(2)}]^{-1}D_2^{(2)}[D_1^{(2)}]^{-1}$.

3. ARE of $\widehat{\theta_n^{(1)}}$ with respect to Hotelling's T^2

In this section we show that $\widehat{\theta_n^{(1)}}$ is equivalent to the Raleigh's statistic (Watson, 1983), whose efficiency was deeply studied by Randles (1989). Let us look at Raleigh's R_n first.

<u>Raleigh's statistic R_n (Watson, 1983)</u>

Let $U_i = \frac{X_i}{\|X_i\|}$ for $i = 1, 2, \cdots, n$. Then $U_1, \cdots, U_n$ are i.i.d. from $Uniform(\Omega_p)$, where Ω_p is the surface of the unit ball in R^p. And the Raleigh's test statistic for the location parameter is defined by

$$R_n \equiv np\ \overline{U}^T\overline{U}.$$ (3.1)

Note that $R_n \to \chi_p^2$ as n goes the infinity. And the first Chaudhuri's statistic $n\ \widehat{\theta_n^{(1)}}^T \Sigma_{U_1}^{-1}\widehat{\theta_n^{(1)}}$ is equivalent to the Raleigh's R_n when the distribution is spherically symmetric as follows:

$$n\ \widehat{\theta_n^{(1)}}^T \Sigma_{U_1}^{-1}\widehat{\theta_n^{(1)}}$$

$$= \sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\frac{X_i^T}{\|X_i\|}[D_1^{(1)}]^{-1}D_1^{(1)}[D_2^{(1)}]^{-1}D_1^{(1)}\sqrt{n}\frac{1}{n}[D_1^{(1)}]^{-1}\sum_{i=1}^{n}\frac{X_i}{\|X_i\|}$$

$$= n\left(\frac{1}{n}\sum_{i=1}^{n}\frac{X_i^T}{\|X_i\|}\right)\left(\frac{1}{p}I_p\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{X_i}{\|X_i\|}\right)$$

$$= np\left(\frac{1}{n}\sum_{i=1}^{n}U_i^T\right)\left(\frac{1}{n}\sum_{i=1}^{n}U_i\right)$$

$$= np \; \overline{U^T U}. \tag{3.2}$$

So we can use the ARE of R_n for $n \; \widehat{\theta_n^{(1)}}^T \Sigma_{U_1}^{-1} \widehat{\theta_n^{(1)}}$.

To look at this ARE more specifically, let us consider a family of elliptically symmetric distributions with density function

$$f(x-\theta) = k_p \; |\Sigma|^{-1/2} \exp(-[x^T \Sigma^{-1} x / c_p]^\nu), \quad x \in R^p, \tag{3.3}$$

where

$$c_p = \frac{p \Gamma\left(\dfrac{p}{2\nu}\right)}{\Gamma\left(\dfrac{p+2}{2\nu}\right)}, \quad k_p = \frac{\nu \Gamma\left(\dfrac{p}{2}\right)}{\Gamma\left(\dfrac{p}{2\nu}\right)[\pi c_p]^{p/2}}, \tag{3.4}$$

Σ denotes a scales matrix and $\nu > 0$. For $0 < \nu < 1$, the distribution is light-tailed, for $\nu = 1$ the distribution is normal, and for $\nu > 1$ the distribution is heavy-tailed. When $\Sigma = I_p$, this family becomes spherically symmetric one.

When observations are i.i.d. from $f(x)$ defined above, based on results by Randles (1989) the asymptotic relative efficiencies of $\widehat{\theta_n^{(1)}}$ with respect to Hotelling's T^2 is obtained such as:

The ARE of $\widehat{\theta_n^{(1)}}$ with respect to Hotelling's T^2 (Randles, 1989)

$$\text{ARE}(\; \widehat{\theta_n^{(1)}}, T^2 \;) = \frac{4\nu^2 \Gamma^2\left(\dfrac{2\nu+p-1}{2\nu}\right)\Gamma\left(\dfrac{p+2}{2\nu}\right)}{p^2 \Gamma^3\left(\dfrac{p}{2\nu}\right)}. \tag{3.5}$$

These AREs are less than 1 if the distribution is light-tailed or normal ($\nu \geq 1$), and they are greater than 1 if the distribution is heavy-tailed ($\nu \leq 1$). This means that $\widehat{\theta_n^{(1)}}$ works better than Hotelling's T^2 when the distribution is light-tailed or normal ($\nu \geq 1$), and Hotelling's T^2 works better than $\widehat{\theta_n^{(1)}}$ when the distribution is heavy-tailed.

4. ARE of $\widehat{\theta_n^{(2)}}$ with respect to Hotelling's T^2

The following proposition provides the ARE of $\widehat{\theta_n^{(2)}}$ with respect to Hotelling's T^2.

Proposition 1.   The asymptotic relative efficiency of $n \; \widehat{\theta_n^{(2)}}^T \Sigma_{U_2}^{-1} \widehat{\theta_n^{(2)}}$ with respect to Hotelling's T^2 is

$$ARE(n \widehat{\theta_n^{(2)T}} \Sigma_{U_2}^{-1} \widehat{\theta_n^{(2)}}, T^2) = E^2 \left[\frac{1}{\|X_1 + X_2\|} \right] \left(\frac{1-p}{p} \right)^2 c^{-1}, \tag{4.1}$$

where $D_2^{(2)} = cI_p$ for a positive c.

Proof. Let us first find the noncentrality parameter Δ of $n \widehat{\theta_n^{(2)T}} \Sigma_{U_2}^{-1} \widehat{\theta_n^{(2)}}$ under the

spherical symmetry. Let $\eta = 2\theta$, $Z = X_1 + X_2$ and $H(Z, \eta) = \frac{Z + \eta/\sqrt{n}}{\|Z + \eta/\sqrt{n}\|}$. Also for the

notational simplicity let us use D_1 and D_2 in place of $D_1^{(2)}$ and $D_2^{(2)}$. From the direct

calculation we obtain the following equation such as

$$\frac{\partial H}{\partial \eta} = \frac{1}{\sqrt{n}} \frac{1}{\|Z + \eta/\sqrt{n}\|} \left(I_p - \frac{Z + \eta/\sqrt{n}}{\|Z + \eta/\sqrt{n}\|} \frac{(Z + \eta/\sqrt{n})^T}{\|Z + \eta/\sqrt{n}\|} \right). \tag{4.2}$$

Then $\frac{\partial H}{\partial \theta} = \frac{\partial H}{\partial \eta} \frac{\partial \eta}{\partial \theta} = 2 \frac{\partial H}{\partial \eta}$. Then $\sqrt{n} E_{H_A} \left[\frac{X_1 + X_2}{\|X_1 + X_2\|} \right]$ are obtained as follows.

$$\sqrt{n} E \left[\frac{X_1 + X_2 + 2\theta/\sqrt{n}}{\|X_1 + X_2 + 2\theta/\sqrt{n}\|} \right]$$

$$\approx \sqrt{n} E \left[\frac{\partial H}{\partial \theta} \Big|_{\theta=0} \right] \theta$$

$$= 2\sqrt{n} E \left[\frac{\partial H}{\partial \eta} \Big|_{\eta=0} \right] \theta$$

$$= 2 E \left[\frac{1}{\|X_1 + X_2\|} \left(I_p - \frac{X_1 + X_2}{\|X_1 + X_2\|} \frac{(X_1 + X_2)^T}{\|X_1 + X_2\|} \right) \right] \theta$$

$$= 2 E \left[\frac{1}{\|X_1 + X_2\|} \right] \left(\frac{p-1}{p} \right) \theta. \tag{4.3}$$

The first approximation holds by Taylor expansion and the last equality holds because $(X_1 + X_2)/\|X_1 + X_2\|$ and $\|X_1 + X_2\|$ are independent under the spherical symmetry. Therefore the noncentrality parameter of $n \widehat{\theta_n^{(2)T}} \Sigma_{U_2}^{-1} \widehat{\theta_n^{(2)}}$ is given by

$$\Delta = 4 E^2 \left[\frac{1}{\|X_1 + X_2\|} \right] \left(\frac{1-p}{p} \right)^2 \theta^T D_1^{-1} [2^2 D_1^{-1} D_2 D_1^{-1}]^{-1} D_1^{-1} \theta$$

$$= E^2 \left[\frac{1}{\|X_1 + X_2\|} \right] \left(\frac{1-p}{p} \right)^2 \theta^T D_2^{-1} \theta$$

$$= E^2 \left[\frac{1}{\|X_1 + X_2\|} \right] \left(\frac{1-p}{p} \right)^2 c^{-1} \theta^T \theta, \tag{4.4}$$

where Chaudhuri (1992) showed that $D_2 = cI_p$ for a positive c under the spherical symmetry. Then its asymptotic relative efficiency follows immediately. $\square$

When X_1, X_2, X_3 are i.i.d. from a spherically symmetric distribution $E[1/\|X_1 + X_2\|]$

and $E\left[\dfrac{X_1 + X_2}{\|X_1 + X_2\|} \dfrac{(X_1 + X_3)^T}{\|X_1 + X_3\|}\right] = cI_p$ for $c > 0$ are two quantities of our interest and they

are $2p$-fold and $3p$-fold integrals respectively. However Choi and Marden (1997) calculated similar expectations of multivariate random variables with Euclidean distances, which are

$E[1/\|X_1 - X_2\|]$ and $E\left[\dfrac{X_1 - X_2}{\|X_1 - X_2\|} \dfrac{(X_1 - X_3)^T}{\|X_1 - X_3\|}\right] = dI_p$ for $d > 0$. They obtained numeric

values of those expectations when X_1, X_2, X_3 are i.i.d. from a spherically symmetric distribution with the density defined in Section 3 for the cases of $p = 1, 2, 3, 5$ and $\nu = 5, 2, 1, 1/2, 1/5, 1/10$. Under the spherical symmetry it is obvious that $E[1/\|X_1 + X_2\|]$ $= E[1/\|X_1 - X_2\|]$ and $c = d$. Therefore the numeric results by Choi and Marden (1997) can be used for Chaudhuri's location estimators.

These AREs are less than 1 if the distribution is light-tailed or normal ($\nu \geq 1$), and they are greater than 1 if the distribution is heavy-tailed ($\nu \leq 1$). This means that $\widehat{\theta_n^{(2)}}$ works better than Hotelling's T^2 when the distribution is light-tailed or normal ($\nu \geq 1$), and Hotelling's T^2 works better than $\widehat{\theta_n^{(2)}}$ when the distribution is heavy-tailed.

5. ARE of $\widehat{\theta_n^{(2)}}$ with respect to $\widehat{\theta_n^{(1)}}$

Once the two AREs are obtained we can compare a seires of Chaudhuri's statistics in terms of AREs, as Chaudhuri did with the variances under the normal distributions. For this, we use the asymptotic relative efficiency of $\widehat{\theta_n^{(2)}}$ with respect to $\widehat{\theta_n^{(1)}}$. This ARE is a simple ratio of two asymptotic relative efficiencies, which is as follows:

$$ARE(\widehat{\theta_n^{(2)}}, \widehat{\theta_n^{(1)}}) = \frac{ARE(\widehat{\theta_n^{(2)}}, T^2)}{ARE(\widehat{\theta_n^{(1)}}, T^2)}. \tag{4.5}$$

When a sample is i.i.d. from the spherically symmetric distribution with the density function defined in Section 3, its numeric values are presented in the following Table 1.

Table 1. The Asymptotic Relative Efficiencies of $\widehat{\theta}_n^{(n)}$ with respect to $\widehat{\theta}_n^{(1)}$

p	ν					
	5	2	1	1/2	1/5	1/10
1	2.614	2.123	1.499	0.750	0.0938	0.00293
2	1.675	1.497	1.232	0.864	0.348	0.105
3	1.409	1.306	1.147	0.911	0.513	0.233
5	1.221	1.169	1.032	0.946	0.659	0.413

Let us first look at the normal distribution case. As mentioned earlier Chaudhuri (1992) proved $\sigma^2(p, L) > \sigma^2(p, L+1)$ which means that $\widehat{\theta}_n^{(L+1)}$ is better than $\widehat{\theta}_n^{(L)}$ under the spherically symmetric normal distributions. This result for $L = 1$ is consistent with the values when $\nu = 1$ in our table. When $\nu = 1$, $ARE(\widehat{\theta}_n^{(2)}, \widehat{\theta}_n^{(1)})$ is always greater than 1. When the distribution is light-tailed ($\nu > 1$), all AREs are also greater than 1. This also means that $\widehat{\theta}_n^{(2)}$ works better than $\widehat{\theta}_n^{(1)}$. On the other hand when the distribution is heavy-tailed ($\nu < 1$), all AREs are less than 1. This means that $\widehat{\theta}_n^{(1)}$ works better than $\widehat{\theta}_n^{(2)}$. For all cases all AREs tend to 1 as dimension increases. Therefore in term of the ARE we conjecture that for the light tailed and normal distribution, $\widehat{\theta}_n^{(n)}$ ($L = n$) is the best and for the heavy tailed, $\widehat{\theta}_n^{(1)}$ ($L = 1$) is the best.

6. Conclusion

Direction vectors have been used to make the test statistics distribution-free in the multivariate samples. Even though statistics with Euclidean norms are easy to calculate and understand, obtaining their efficiencies are very difficult because expectations of Euclidean norms often involve integrations in high dimensions. Based on Euclidean distances, Chaudhuri extended univariate Hodges-Lehmann location estimators to the multivariate ones. In this paper we obtained exact AREs of two simplest Chaudhuri's test statistics and compared them in terms of their asymptotic efficiencies. To obtain numeric values of AREs we used results by Randles(1989) and Choi and Marden(1997). When the distribution is heavy-tailed, the average direction vector of each observation is better than the average direction vector of sum of two observations. When the distribution is light-tailed or normal, it is the other way around.

It would be also good to have some simulation results for the comparison. For this we can refer to the simulation results by Randles (1989) and by Choi and Marden (1997). Even though Randles' test statistic is not exactly the same as Raleigh's one, they are asymptotically equivalent. Randles uses the interdirections to estimate the angles between two unit vectors and he shows that these interdirections provide the consistent estimators of inner products of the unit vectors. So the simulation results for Raleigh's would not be different from those for Randles'. Also we can give the similar reasons for Chaudhuri's second location estimators. Even though the statistic of Choi and Marden (1997) is for the two-sample problem, its empirical behavior with respect to the two-sample Hotelling's T^2 would not be very much different from that of Chaudhuri's second one with respect to the one-sample Hotelling's T^2. This is because both test statistics use the simple average of direction vectors of two observations and have the same efficiencies. Thus the simulation results of the second Chaudhuri's statistic would not be really different from that of Choi and Marden's one.

Also it is another task to have a proper consistent variance-covariance estimators of Chaudhuri's location estimators.

References

[1]   Bickel, P.J. (1977) *Mathematical Statistics*, Holden-Day, Inc.

[2]   Chaudhuri, P. (1992) Multivariate location estimation using extension of R-estimates through U-statistics type approach. *Annals of Statistics*, 20, no.2, 897-916.

[3]   Chaudhuri, P. and Sengupta, D.(1993) Sign tests in multidimension : Inference based on the geometry of the data cloud. *Journal American Statistical Association*, 88, 1363-1370.

[4]   Choi, K. (1997) Breakdown Points of Direction Tests. *J. Korean Statistical Society*, 26, no.2, 211-222.

[5]   Choi, K. and Marden, J. I. (1997) An approach to multivariate rank tests in multivariate analysis of variance. *Journal of American Statistical Association*, 92, 1581-1590.

[6]   Hodges, J. L. and Lehmann, E. L. (1963) Estimates of location based on rank tests. *Annals of Mathematical Statistics*, 34, 598-611.

[7]   Kim, D. H., Kim, Y. C., and Cho, M. H. (1996) Distribution-Free Two-Sample Test on Ranked-Set Samples. *The Korean Communications in Statistics*, 5, no.1, 133-144.

[7]   Mardia, K. V. et al. (1979) *Multivariate Analysis*, Academic Press.

[8]   Randles,R.H. (1989) A distribution-free multivariate sign test based on interdirections. *Journal of American Statistical Association*, 84, 1045-1050.

[9]   Watson, G. S. (1983) *Statistics on Spheres*, Wiley-Interscience, New York.