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SOME CONDITIONS ON DERIVATIONS IN PRIME NEAR-RINGS

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ABSTRACT. Posner [*Proc. Amer. Math. Soc.* **8** (1957), 1093–1100] defined a derivation on prime rings and Herstein [*Canad. Math. Bull.* **21** (1978), 369–370] derived commutative property of prime ring with derivations. Recently, Bergen [*Canad. Math. Bull.* **26** (1983), 267–227], Bell and Daif [*Acta. Math. Hungar.* **66** (1995), 337–343] studied derivations in primes and semiprime rings. Also, in near-ring theory, Bell and Mason [*Near-Rings and Near-Fields* (pp. 31–35), Proceedings of the conference held at the University of Tübingen, 1985. Noth-Holland, Amsterdam, 1987; *Math. J. Okayama Univ.* **34** (1992), 135–144] and Cho [Pusan Kyongnam Math. J. **12** (1996), no. 1, 63–69] researched derivations in prime and semiprime near-rings. In this paper, Posner, Bell and Mason's results are extended in prime near-rings with some conditions.

1. INTRODUCTION

Throughout this paper, N will denote a zero-symmetric left near-ring. A near-ring N is called a *prime near-ring* if N has the property that for $a,\ b \in N,\ aNb = \{0\}$ implies $a = 0$ or $b = 0$. N is called a *semiprime near-ring* if N has the property that for $a \in N,\ aNa = \{0\}$ implies $a = 0$. A nonempty subset U of N is called a *right N-subset* (resp. *left N-subset*) if $UN \subset U$ (resp. $NU \subset U$), and if U is both a right N-subset and a left N-subset, it is said to be an *N-subset* of N.

An *ideal* of N is a subset I of N such that

(i) $(I,+)$ is a normal subgroup of $(N,+)$,
(ii) $a(I+b) - ab \subset I$ for all $a,\ b \in N$,
(iii) $(I+a)b - ab \subset I$ for all $a,\ b \in N$.

If I satisfies (i) and (ii) then it is called a *left ideal* of N. If I satisfies (i) and (iii) then it is called a *right ideal* of N.

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On the other hand, a (two-sided) N-*subgroup* of N is a subset H of N such that

(i) $(H, +)$ is a subgroup of $(N, +)$,

(ii) $NH \subset H$, and

(iii) $HN \subset H$.

If H satisfies (i) and (ii) then it is called a *left* N-*subgroup* of N. If H satisfies (i) and (iii) then it is called a *right* N-*subgroup* of N. Note that normal N-subgroups of N are not equivalent to ideals of N.

Every right ideal of N, right N-subgroup of N and right semigroup ideal of N are right N-subsets of N, and symmetrically, we can apply for the left case. A *derivation* D on N is an additive endomorphism of N with the property that for all $a,\ b \in N$, $D(ab) = aD(b) + D(a)b$.

In ring theory, In 1957, Posner [9] defined a derivation on prime rings and in 1978, Herstein [6] derived commutative property of prime rings with derivations. Recently, Bergen [4], Bell and Daif [1] studied derivations in prime and semiprime rings, and commutativity of prime rings with derivations. Also, in near-ring theory, Bell and Mason in 1987 [2], in 1992 [3] and Cho [5] researched derivations in prime and semiprime near-rings. In this paper, Posner, Bell and Mason's results are slightly extended in prime near-rings with some conditions.

All other basic properties, terminologies and concepts are appeared in the books of Meldrum [7] and Pilz [8].

2. Conditions On Derivations In Prime Near-Rings

A near-ring N is called *abelian* if $(N, +)$ is abelian, and *2-torsion free* if for all $a \in N$, $2a = 0$ implies $a = 0$.

Lemma 2.1. *Let D be an arbitrary additive endomorphism of N. Then*

$$D(ab) = aD(b) + D(a)b \text{ if and only if } D(ab) = D(a)b + aD(b)$$

for all $a,\ b \in N$.

Proof. Suppose that $D(ab) = aD(b)+D(a)b$, for all $a, b \in N$. From $a(b+b) = ab+ab$ and N satisfies left distributive law

$$\begin{aligned} D(a(b+b)) &= aD(b+b) + D(a)(b+b) \\ &= a(D(b) + D(b)) + D(a)b + D(a)b \\ &= aD(b) + aD(b) + D(a)b + D(a)b \end{aligned}$$

and

$$D(ab + ab) = D(ab) + D(ab) = aD(b) + D(a)b + aD(b) + D(a)b.$$

Comparing these two equalities, we have $aD(b) + D(a)b = D(a)b + aD(b)$. Hence $D(ab) = D(a)b + aD(b)$.

Conversely, suppose that $D(ab) = D(a)b + aD(b)$, for all $a, b \in N$. Then from $D(a(b + b)) = D(ab + ab)$ and the above calculation of this equality, we can induce that $D(ab) = aD(b) + D(a)b$. □

Lemma 2.2. [5] *Let D be a derivation on N. Then N satisfies the following right distributive laws: for all a, b, c in N,*

$$\{aD(b) + D(a)b\}c = aD(b)c + D(a)bc,$$

$$\{D(a)b + aD(b)\}c = D(a)bc + aD(b)c.$$

Proof. From the calculation for $D((ab)c) = D(a(bc))$ and Lemma 2.1, we can induce our result. □

Lemma 2.3. *Let N be a prime near-ring and let U be a nonzero N-subset of N. If x be an element of N such that $Ux = \{0\}$ (or $xU = \{0\}$), then $x = 0$.*

Proof. Since $U \neq \{0\}$, there exist an element $u \in U$ such that $u \neq 0$.

Consider that $uNx \subset Ux = \{0\}$. Since $u \neq 0$ and N is a prime near-ring, we have that $x = 0$. □

Corollary 2.4. *Let N be a semiprime near-ring and let U be a nonzero N-subset of N. If x be an element of $N(U)$ such that $Ux^2 = \{0\}$ (or $x^2U = \{0\}$), where $N(U)$ is the normalizer of U, then $x = 0$.*

Lemma 2.5. *Let N be prime and U a nonzero N-subset of N. If D is a nonzero derivation on N. Then*

(i) *If $a, b \in N$ and $aUb = \{0\}$, then $a = 0$ or $b = 0$.*
(ii) *If $a \in N$ and $D(U)a = \{0\}$, then $a = 0$.*
(iii) *If $a \in N$ and $aD(U) = \{0\}$, then $a = 0$.*

Proof. (i) Let $a, b \in N$ and $aUb = \{0\}$. Then $aUNb \subset aUb = \{0\}$. Since N is a prime near-ring, $aU = 0$ or $b = 0$.

If $b = 0$, then we are done. So if $b \neq 0$, then $aU = 0$. Applying Lemma 2.3, $a = 0$.

(ii) Suppose $D(U)a = \{0\}$, for $a \in N$. Then for all $u \in U$ and $b \in N$, from Lemma 2.2, we have

$$0 = D(bu)a = (bD(u) + D(b)u)a = bD(u)a + D(b)ua = D(b)ua.$$

Hence $D(b)Ua = \{0\}$ for all $b \in N$. Since D is a nonzero derivation on N, we have that $a = 0$ by the statement (i).
(iii) Suppose $aD(U) = \{0\}$ for $a \in N$. Then for all $u \in U$ and $b \in N$,

$$0 = aD(ub) = a\{uD(b) + D(u)b\} = auD(b) + aD(u)b = auD(b).$$

Hence $aUD(b) = \{0\}$ for all $b \in N$. From the statement (i) and D is a nonzero derivation on N, we have that $a = 0$. □

We remark that to obtain any of the conclusions of Lemma 2.5, it is not sufficient to assume that U is a right N-subset, even in the case that N is a ring. Consider the following example:

Example 2.6. Let R be the prime ring $Mat_2(F)$, where F is an arbitrary field. Let

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R$$

and let D be the inner derivation of R given by

$$D(w) = w \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} w.$$

Then

$$D(U) = \{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} R \mid a \in F\},$$

so that for

$$x = y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

we have

$$xUy = xD(u) = D(U)x = \{0\}.$$

Theorem 2.7. *Let N be a prime near-ring and U a right N-subset of N. If D is a nonzero derivation on N such that $D^2(U) = 0$, then $D^2 = 0$.*

Proof. For all $u, v \in U$, we have $D^2(uv) = 0$. Then

$$\begin{aligned} 0 = D^2(uv) = D(D(uv)) &= D\{D(u)v + uD(v)\} \\ &= D^2(u)v + D(u)D(v) + D(u)D(v) + uD^2(v) \\ &= D^2(u)v + 2D(u)D(v) + uD^2(v). \end{aligned}$$

Thus $2D(u)D(U) = \{0\}$ for all $u \in U$. From Lemma 2.5 (iii), we have $2D(u) = 0$. Now for all $b \in N$ and $u \in U$, $D^2(ub) = uD^2(b) + 2D(u)D(b) + D^2(u)b$.

Hence $UD^2(b) = \{0\}$ for all $b \in N$. By Lemma 2.3, we have $D^2(b) = 0$ for all $b \in N$. Consequently $D^2 = 0$. □

Lemma 2.8. *Let D be a derivation of a prime near-ring N and a be an element of N. If $aD(x) = 0 (or D(x)a = 0)$ for all $x \in N$, then either $a = 0$ or D is zero.*

Proof. Suppose that $aD(x) = 0$ for all $x \in N$. Replacing x by xy, we have that $aD(xy) = 0 = aD(x)y + axD(y)$ by Lemma 2.2. Then $axD(y) = 0$ for all $x, y \in N$.

If D is not zero, that is, if $D(y) \neq 0$ for some $y \in N$, then, since N is a prime near-ring, $aND(y)$ implies that $a = 0$. □

Now we prove our main result, which extends a famous theorem on rings of Posner [9] to near-rings with some condition.

Theorem 2.9. *Let N be a prime near-ring with nonzero derivations D_1 and D_2 such that for all $x, y \in N$,*

$$D_1(x)D_2(y) = -D_2(x)D_1(y) \tag{1}$$

Then N is an abelian near-ring.

Proof. Let $x, u, v \in N$. From the condition (1), we obtain that

$$\begin{aligned} 0 &= D_1(x)D_2(u+v) + D_2(x)D_1(u+v) \\ &= D_1(x)[D_2(u) + D_2(v)] + D_2(x)[D_1(u) + D_1(v)] \\ &= D_1(x)D_2(u) + D_1(x)D_2(v) + D_2(x)D_1(u) + D_2(x)D_1(v) \\ &= D_1(x)D_2(u) + D_1(x)D_2(v) - D_1(x)D_2(u) - D_1(x)D_2(v) \\ &= D_1(x)[D_2(u) + D_2(v) - D_2(u) - D_2(v)] \\ &= D_1(x)D_2(u + v - u - v). \end{aligned}$$

Thus

$$D_1(N)D_2(u + v - u - v) = \{0\}. \tag{2}$$

By Lemma 2.8, we have

$$D_2(u + v - u - v) = 0. \tag{3}$$

Now, we substitute xu and xv instead of u and v respectively in (3). Then from Lemma 2.1, we deduce that for all $x, u, v \in N$,

$$\begin{aligned}0 &= D_2(xu + xv - xu - xv)\\ &= D_2[x(u + v - u - v)]\\ &= D_2(x)(u + v - u - v) + xD_2(u + v - u - v)\\ &= D_2(x)(u + v - u - v).\end{aligned}$$

Again, applying Lemma 2.8, we see that for all $u, v \in N$,

$$u + v - u - v = 0.$$

Consequently, N is an abelian near-ring. □

Theorem 2.10. *Let N be a prime near-ring of 2-torsion free and let D_1 and D_2 be derivations with the condition*

$$D_1(a)D_2(b) = D_2(b)D_1(a) \tag{4}$$

for all $a, b \in N$ on N. Then D_1D_2 is a derivation on N if and only if either $D_1 = 0$ or $D_2 = 0$

Proof. Suppose that D_1D_2 is a derivation. Then we obtain

$$D_1D_2(ab) = aD_1D_2(b) + D_1D_2(a)b. \tag{5}$$

Also, since D_1 and D_2 are derivations, we get

$$\begin{aligned}D_1D_2(ab) &= D_1(D_2(ab)) = D_1(aD_2(b) + D_2(a)b)\\ &= D_1(aD_2(b)) + D_1(D_2(a)b)\\ &= aD_1D_2(b) + D_1(a)D_2(b) + D_2(a)D_1(b) + D_1D_2(a)b. \end{aligned}\tag{6}$$

From (5) and (6) for $D_1D_2(ab)$ for all $a, b \in N$,

$$D_1(a)D_2(b) + D_2(a)D_1(b) = 0. \tag{7}$$

Hence from Theorem 2.9, we know that N is an abelian near-ring.

Replacing a by $aD_2(c)$ in (7), and using Lemma 2.1 and Lemma 2.2, we obtain that

$$\begin{aligned}0 &= D_1(aD_2(c))D_2(b) + D_2(aD_2(c))D_1(b)\\ &= \{D_1(a)D_2(c) + aD_1D_2(c)\}D_2(b) + \{aD_2^2(c) + D_2(a)D_2(c)\}D_1(b)\\ &= D_1(a)D_2(c)D_2(b) + aD_1D_2(c)D_2(b) + aD_2^2(c)D_1(b) + D_2(a)D_2(c)D_1(b)\\ &= D_1(a)D_2(c)D_2(b) + a\{D_1D_2(c)D_2(b) + D_2^2(c)D_1(b)\} + D_2(a)D_2(c)D_1(b).\end{aligned}$$

On the other hand, replacing a by $D_2(c)$ in (7), we see that

$$D_1(D_2(c))D_2(b) + D_2(D_2(c))D_1(b) = 0.$$

This equation implies that

$$a\{D_1D_2(c)D_2(b) + D_2^2(c)D_1(b)\} = 0.$$

Hence, from the above last long equality, we have the following equality:

$$D_1(a)D_2(c)D_2(b) + D_2(a)D_2(c)D_1(b) = 0, \tag{8}$$

for all $a, b, c \in N$. Replacing a and b by c in (7) respectively, we see that

$$D_2(c)D_1(b) = -D_1(c)D_2(b),$$
$$D_1(a)D_2(c) = -D_2(a)D_1(c).$$

So that (8) becomes

$$\begin{aligned} 0 &= \{-D_2(a)D_1(c)\}D_2(b) + D_2(a)\{-D_1(c)D_2(b)\} \\ &= D_2(a)(-D_1(c))D_2(b) + D_2(a)(-D_1(c))D_2(b) \\ &= D_2(a)\{(-D_1(c))D_2(b) - D_1(c)D_2(b)\} \end{aligned}$$

for all $a, b, c \in N$. If $D_2 \neq 0$, then by Lemma 2.8, we have the equality:

$$(-D_1(c))D_2(b) - D_1(c)D_2(b) = 0,$$

that is,

$$D_1(c)D_2(b) = (-D_1(c))D_2(b) \tag{9}$$

for all $b, c \in N$.

Thus, using the given condition of our theorem, we get

$$\begin{aligned} (-D_1(c))D_2(b) = D_1(-c)D_2(b) = D_2(b)D_1(-c) &= D_2(b)(-D_1(c)) \\ &= -D_2(b)D_1(c) = -D_1(c)D_2(b). \end{aligned} \tag{10}$$

From (9) and (10) we have that, for all $b, c \in N$,

$$2D_1(c)D_2(b) = 0.$$

Since N is of 2-torsion free, $D_1(c)D_2(b) = 0$. Also, since D_2 is not zero, by Lemma 2.8, we see that $D_1(c) = 0$ for all $c \in N$. Therefore $D_1 = 0$. Consequently, either $D_1 = 0$ or $D_2 = 0$.

The converse verification is obvious. Thus our proof is complete. □

As a consequence of Theorem 2.10, we get the following important statement.

Corollary 2.11. *Let N be a prime near-ring of 2-torsion free, and let D be a derivation on N such that $D^2 = 0$. Then $D = 0$.*

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