# Asymmetric Least Squares Estimation for A Nonlinear Time Series Regression Model

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## Abstract

The least squares method is usually applied when estimating the parameters in the regression models. However the least square estimator is not very efficient when the distribution of the error is skewed. In this paper, we propose the asymmetric least square estimator for a particular nonlinear time series regression model, and give the simple and practical sufficient conditions for the strong consistency of the estimators.

*Keywords* : Asymmetric least squares estimators, Strong consistency, Nonlinear time series regression models.

## 1. Introduction

Generally, the nonlinear regression model is

$$y_t = f(x_t, \theta_0) + \varepsilon_t, \quad t = 1, 2, \ldots, n$$

where $f(x_t, \theta_0)$ is a real valued nonlinear function defined on $R^{p_1 + p_2}$, $x_t$ is a $(1 \times p_1)$ observed vector, the error term $\varepsilon_t$ are independent and identically distributed (i.i.d.) with zero mean and finite variance. The parameter vector $\theta_0$ which is interior point in a parameter space $\Theta \subset R^{p_2}$ is unknown and to be estimated.

The least squares method still plays a central role in the estimation of parameter $\theta_0$ in the nonlinear regression models. The nonlinear least squares estimation problem is defined as : Find the parameters which minimize

$$Q_n(\theta) = \sum_{t=1}^{n} (y_t - f(x_t, \theta))^2.$$

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A measurable function $\widetilde{\theta}_n$ for which $Q_n(\widetilde{\theta}_n) = \inf_{\theta \in \overline{\Theta}} Q_n(\theta)$ is called a least squares estimator (LSE) for the parameter $\theta$ based on the observations $y_t$, $t = 1, 2, \ldots, n$. Here $\overline{\Theta}$ denotes the closure of the set $\Theta$. Jennrich(1969) first rigorously proved the existence and measurability of the LSE and showed the consistency and asymptotic normality of the LSE $\widetilde{\theta}_n$ under the several assumptions. Wu(1981) gave some sufficient conditions such as a Lipschitz type condition on the sequence $f(x_t, \theta)$ to prove the asymptotic properties of the LSE $\widetilde{\theta}_n$.

The concept of periodicity in time series is of fundamental interest, since it provides a means for formalizing the notions of dependence or correlation between adjacent points. In this paper we think about a sum of sinusoidal components :

$$f(x_t, \theta_0) = \sum_{t=1}^{n} \{A_{r0}\cos(\omega_{r0}t) + B_{r0}\sin(\omega_{r0}t)\},$$

where $\theta_0 = (A_{10}, B_{10,} \omega_{10}, \cdots, A_{q0}, B_{q0}, \omega_{q0})$ for $q \geq 1$, $A_{r0}, B_{r0}$'s are some fixed unknown constants, is unknown frequency lying between 0 to $\pi$ $(1 \leq r \leq q)$ and in this case the observed value $x_t$ means $t$. But the above formula does not satisfy Jennrich(1969)'s assumption nor Wu(1981)'s Lipschitz type condition, the previous method to gain the LSE is not available. Walker(1971) obtained the asymptotic properties of an approximate LSE. Recently, Kundu(1993) and Kundu and Mitra(1996) gave the direct proof of consistency of the LSE.

On the other hand, in spite of the theoretical and practical merits, a certain criticisms of procedures based on the least squares methods in the past have been pointed to the robustness even a single outlier of a slighter departure from the normality assumption on the errors. When the distribution of error is skewed, the LSE is judged inadequate. In many applied regression problem the distribution of errors is skewed, hence an alternative to ordinary LSE methods such as asymmetric least squares( ALS ) method is considered (Newey & Powell 1987).

For that reason, we study the ALS which will be defined in (1.2) of the following nonlinear time series regression model with some assumptions,

(1.1) $$y_t = \sum_{r=1}^{q} [A_{r0}\cos(\omega_{r0}t) + B_{r0}\sin(\omega_{r0}t)] + \varepsilon_t.$$

The ALS of the true parameter $\theta_0 = (A_{10}, B_{10}, \omega_{10}, \cdots, A_{q0}, B_{q0}, \omega_{q0})$ denoted by $\widehat{\theta}_n = (\widehat{A_{1n}}, \widehat{B_{1n}}, \widehat{\omega_{1n}}, \cdots, \widehat{A_{qn}}, \widehat{B_{qn}}, \widehat{\omega_{qn}})$ is a vector value which minimizes the objective function

(1.2) $$R_n(\theta ; \tau) = \frac{1}{n} \sum_{t=1}^{n} \rho_\tau \left( y_t - \sum_{r=1}^{q} [A_r\cos(\omega_r t) + B_r\sin(\omega_r t)] \right),$$

where $0 < \tau < 1$, $\theta = (A_1, B_1, \omega_1, \cdots, A_q, B_q, \omega_q)$ and $\rho_\tau$ is a check function which is defined

by the formula

$$\rho_\tau(\lambda) = |\tau - I_{\{\lambda<0\}}|\lambda^2, \quad I_{\{\lambda<0\}} \text{ is the indicate function}$$
$$= \begin{cases} (1-\tau)\lambda^2, & \lambda < 0 \\ \tau\lambda^2, & \lambda \geq 0. \end{cases}$$

In this paper, we will study the strong consistency of the Asymmetric Least Squares Estimators for the nonlinear time series regression model (1.1).

# 2. The Strong Consistency

Firstly, for the case $q = 1$, i.e. for the one harmonic component, we will consider the strong consistency of the nonlinear ALS $\widehat{\theta_n} = \widehat{\theta_{1n}} = (\widehat{A_{1n}}, \widehat{B_{1n}}, \widehat{\omega_{1n}}) = (\widehat{A_n}, \widehat{B_n}, \widehat{\omega_n})$ for $\theta_0 = (A_{10}, B_{10}, \omega_{10}) = (A_0, B_0, \omega_0)$ in a time series with stationary independent residuals model (1.1) with the following assumptions.

### Assumption A

The parameter space $\Theta = K \times K \times [0, \pi]$, where $K$ is compact subspace of $R$.

### Assumption B

**B1:** $\{\varepsilon_t\}$ are i.i.d. continuous random variables which have distribution function $G(\varepsilon_t)$ and probability density function $g(\varepsilon_t)$.

**B2:** $E(\varepsilon_t^2) < \infty$ for all $t$.

**B3:** We define $\tau$ as

$$(1-\tau)\int_{-\infty}^0 \varepsilon_t g(\varepsilon_t)\, d\varepsilon_t + \tau \int_0^\infty \varepsilon_t g(\varepsilon_t)\, d\varepsilon_t = 0,$$

where $\int dG(\varepsilon_t) = \int g(\varepsilon_t)\, d(\varepsilon_t)$.

Note that in B3, if we define $a = E(\varepsilon_t)$, $b = \int_{-\infty}^0 \varepsilon_t g(\varepsilon_t)\, d\varepsilon_t$, $c = \int_0^\infty \varepsilon_t g(\varepsilon_t)\, d\varepsilon_t$, then

$$(1-\tau)b + \tau c = (1-\tau)b + \tau(a-b) = 0, \quad \tau = \frac{b}{2b-a} = \frac{b}{b-c}.$$

Now we define the new object function :

$$Q_n(\theta;\tau) = R_n(\theta;\tau) - R_n(\theta_0;\tau).$$

Since $R_n(\theta_0;\tau)$ is constant for $\theta$, so if we define $\widehat{\theta_\tau}$ as a minimizer of $Q_n(\theta;\tau)$, it is also a minimizer of $R_n(\theta;\tau)$.

The main tools in the proof of the strong consistency of ALS $\widehat{\theta_n}$ are the following two lemmas.

**Lemma 2.1** *Suppose that Assumptions A and B are satisfied on the model (1.1).*
*Then* $Q_n(\theta;\tau)$ *converges to* $Q(\theta;\tau)$ *uniformly for all* $\theta$ *in* $\Theta$ *and almost surely, where*
$$Q(\theta;\tau) = \lim_{n\to\infty} E[Q_n(\theta;\tau)].$$

**Proof** : A proof of Lemma 2.1 is included in Appendix.

**Lemma 2.2** *Suppose that Assumptions A and B are satisfied on the model (1.1).*
*Then* $Q(\theta;\tau)$ *has the unique minimizer* $\theta_0$ *in* $\Theta$ .
**Proof** : Also, à proof of Lemma 2.2 is included in Appendix.

From the results of Lemma 2.1 and Lemma 2.2, we conclude the following main result.

**Theorem 2.3** *Under the same conditions of the Lemma 2.1 and Lemma 2.2, we have* $\widehat{\theta}_n$
*converges to* $\theta_0$ *almost surely.*

**Proof** : The above Lemma 2.1 and Lemma 2.2 are the sufficient conditions of White(1980)
for the strong consistency of estimators, we know that $\widehat{\theta}_n$ converges to $\theta_0$ almost surely.

Also for several harmonic component case, i.e. $q > 1$ , by the same process in Lemma 2.1
and Lemma 2.2, we found out $Q_n(\theta;\tau)$ converges to $Q(\theta;\tau)$ almost surely, and  at least
$\theta_0$ is a local minimizer of $Q(\theta;\tau)$. But for the several harmonic components, we need some
additional condition. It must be imposed to keep the $\omega_r$ from being too close together and thus
prevent estimators of two angular frequencies from converging in probability to the same
value. So the required condition is

(2.1) $$\lim_{n\to\infty} \min_{1\le r\ne s\le q} (n|\omega_r - \omega_s|) = \infty$$

we can also prove the fact $Q(\theta;\beta) > 0$, for $\theta \ne \theta_0$, and $Q(\theta_0;\tau) = 0$, by the same
calculation for the $q = 1$ . Therefore we can state the result as the following Theorem.

**Theorem 2.4** *Under the same assumptions of Theorem 2.3 and with the condition (2.1),*
*the ALS* $\widehat{\theta}_n$ *is a strongly consistent estimator of* $\theta_0$.

# Appendix ; Proofs of Lemmas

**Proof of Lemma 2.1** : We can rewrite the new object function $Q_n(\theta;\tau)$ as follows
$$Q_n(\theta;\tau) = R_n(\theta;\tau) - R_n(\theta_0;\tau)$$

$$= \frac{1}{n} \sum_{t=1}^{n} \left[ |\tau - I_{\{y_t < Acos\omega t + Bsin\omega t\}}| \{y_t - (Acos\omega t + Bsin\omega t)\}^2 \right.$$
$$\left. - |\tau - I_{\{y_t < A_0 \cos\omega_0 t + B_0 \sin\omega_0 t\}}| \{y_t - (A_0 \cos\omega_0 t + B_0 \sin\omega_0 t)\}^2 \right]$$
$$= \frac{1}{n} \sum_{t=1}^{n} \left[ |\tau - I_{\{\varepsilon_t < d_t(\theta)\}}| (\varepsilon_t - d_t(\theta))^2 - |\tau - I_{\{\varepsilon_t < 0\}}| \varepsilon_t^{\,2} \right],$$

where $d_t(\theta) = (Acos\omega t + Bsin\omega t) - (A_0 \cos\omega_0 t + B_0 \sin\omega_0 t)$. And we let

$$X_t = |\tau - I_{\{\varepsilon_t < d_t(\theta)\}}| (\varepsilon_t - d_t(\theta))^2 - |\tau - I_{\{\varepsilon_t < 0\}}| \varepsilon_t^2,$$

then

$$(A.1) \quad X_t = \begin{cases} \tau(2\varepsilon_t - d_t(\theta))(-d_t(\theta)), & \varepsilon_t \geq d_t(\theta),\ \varepsilon_t \geq 0 \\ \tau d_t^2(\theta) - \tau 2\varepsilon_t d_t(\theta) + (2\tau - 1)\varepsilon_t^2, & \varepsilon_t \geq d_t(\theta),\ \varepsilon_t < 0 \\ (1-\tau) d_t^2(\theta) - (1-\tau) 2\varepsilon_t d_t(\theta) + (1-2\tau)\varepsilon_t^2, & \varepsilon_t < d_t(\theta),\ \varepsilon_t \geq 0 \\ (1-\tau)(2\varepsilon_t - d_t(\theta))(-d_t(\theta)), & \varepsilon_t < d_t(\theta),\ \varepsilon_t < 0. \end{cases}$$

Thus

$$|X_t| \leq |\varepsilon_t - d_t(\theta)||d_t(\theta)| \leq (|\varepsilon_t| + |d_t(\theta)|)|d_t(\theta)|$$

or

$$|X_t| \leq |d_t(\theta)|^2 + 2|\varepsilon_t||d_t(\theta)| + |\varepsilon_t(\theta)|^2.$$

Also, we know that $E(\varepsilon_t^2) < \infty$ by assumption B2. Thus $|X_t| < \infty$ and $E(X_t) < \infty$. Using Kolmogorov's SLLN, we conclude that

$$\frac{\sum_{t=1}^{n} X_t - \sum_{t=1}^{n} E(X_t)}{n} \rightarrow 0 \quad \text{almost surely, as} \quad n \rightarrow \infty.$$

Therefore, we have $Q_n(\theta;\tau)$ converges to $Q(\theta;\tau)$ almost surely.

**Proof of Lemma 2.2 :**

We can reduce the form of $E(X_t)$ as follows :

$$E(X_t) = (1-\tau) \left[ \int_{-\infty}^{d_t(\theta)} \varepsilon_t^2 dG(\varepsilon_t) - \int_{-\infty}^{0} \varepsilon_t^2 dG(\varepsilon_t) - 2d_t(\theta) \int_{-\infty}^{d_t(\theta)} \varepsilon_t dG(\varepsilon_t) \right.$$
$$\left. + d_t^2(\theta) \int_{-\infty}^{d_t(\theta)} dG(\varepsilon_t) \right]$$
$$+ \tau \left[ \int_{d_t(\theta)}^{\infty} \varepsilon_t^2 dG(\varepsilon_t) - \int_{0}^{\infty} \varepsilon_t^2 dG(\varepsilon_t) - 2d_t(\theta) \int_{d_t(\theta)}^{\infty} \varepsilon_t dG(\varepsilon_t) \right.$$
$$\left. + d_t^{2}(\theta) \int_{d_t(\theta)}^{\infty} dG(\varepsilon_t) \right].$$

If $d_t(\theta) > 0$ then we obtain the following formula :

$$E(X_t) = (1-\tau)\left[\int_0^{d_t(\theta)} \varepsilon_t^2 dG(\varepsilon_t) - 2d_t(\theta)\int_{-\infty}^{d_t(\theta)} \varepsilon_t dG(\varepsilon_t) + d_t^2(\theta)G(d_t(\theta))\right]$$
$$+ \tau\left[-\int_0^{d_t(\theta)} \varepsilon_t^2 dG(\varepsilon_t) - 2d_t(\theta)\int_{d_t(\theta)}^{\infty} \varepsilon_t dG(\varepsilon_t) + d_t^2(\theta)\{1 - G(d_t(\theta))\}\right].$$

Using the Mean Value Theorem,

$$\int_0^{d_t(\theta)} \varepsilon_t^2 g(\varepsilon_t)d(\varepsilon_t) = d_t^{*2}(\theta)g(d_t^*(\theta))d_t(\theta) < \infty \quad \text{for some} \quad d_t^*(\theta) \in [0, d_t(\theta)].$$

On the other hand,

$$(1-\tau)\int_{-\infty}^{d_t(\theta)} \varepsilon_t dG(\varepsilon_t) + \tau\int_{d_t(\theta)}^{\infty} \varepsilon_t dG(\varepsilon_t)$$

$$= (1-\tau)\int_{-\infty}^{d_t(\theta)} \varepsilon_t dG(\varepsilon_t) + \tau\left[\int_{-\infty}^{\infty} \varepsilon_t dG(\varepsilon_t) - \int_{-\infty}^{d_t(\theta)} \varepsilon_t dG(\varepsilon_t)\right]$$

$$= (1-2\tau)\int_{-\infty}^{d_t(\theta)} \varepsilon_t dG(\varepsilon_t) + \tau E(\varepsilon_t)$$

$$= (1-2\tau)\{K(d_t(\theta)) - K(-\infty)\} + \tau E(\varepsilon_t),$$

where $K$ is the indefinite integral of $\varepsilon_t g(\varepsilon_t)$. Since $E(\varepsilon_t^2) < \infty$, we have

$$E(X_t) = (1-2\tau)\left[d_t^{*2}(\theta)g(d_t^*(\theta))d_t(\theta) + d_t^2(\theta)G(d_t(\theta))\right.$$
$$\left. - 2d_t(\theta)\{K(d_t(\theta)) - K(-\infty)\}\right]$$
$$+ \tau d_t^2(\theta) - 2d_t(\theta)\tau E(\varepsilon_t)$$
$$< \infty.$$

For the case of $d_t(\theta) < 0$,

$$(1-\tau)\int_{-\infty}^{d_t(\theta)} \varepsilon_t dG(\varepsilon_t) + \tau\int_{d_t(\theta)}^{\infty} \varepsilon_t dG(\varepsilon_t)$$

$$= (1-\tau)\int_{-\infty}^{0} \varepsilon_t dG(\varepsilon_t) + \tau\int_0^{\infty} \varepsilon_t dG(\varepsilon_t) + (1-2\tau)\int_0^{d_t(\theta)} \varepsilon_t dG(\varepsilon_t).$$

Also, using the Mean Value Theorem, for some $d_t^{**}(\theta) \in [d_t(\theta), 0]$, we have the same result $E(X_t) < \infty$.

Using the fact that

$$\frac{\partial}{\partial A} d_t(\theta) = \cos \omega t, \quad \frac{\partial}{\partial B} d_t(\theta) = \sin \omega t, \quad \frac{\partial}{\partial \omega} d_t(\theta) = -At\sin\omega t + Bt\cos\omega t,$$

then we have followings :

$$\frac{\partial}{\partial A} G(d_t(\theta)) = g(d_t(\theta))\cos\omega t, \quad \frac{\partial}{\partial B} G(d_t(\theta)) = g(d_t(\theta))\sin\omega t,$$

$$\frac{\partial}{\partial \omega} G(d_t(\theta)) = g(d_t(\theta))(-At\sin\omega t + Bt\cos\omega t),$$

and

$$\frac{\partial}{\partial A} K(d_t(\theta)) = d_t(\theta)g(d_t(\theta))\cos\omega t, \quad \frac{\partial}{\partial B} K(d_t(\theta)) = d_t(\theta)g(d_t(\theta))\sin\omega t,$$

$$\frac{\partial}{\partial \omega} K(d_t(\theta)) = d_t(\theta)g(d_t(\theta))(-At\sin\omega t + Bt\cos\omega t).$$

So we gain followings :

$$\frac{\partial}{\partial A} E[Q_n(\theta;\tau)] = \frac{1}{n} \sum_{t=1}^{n} \cos \omega t \{(1-2\tau)[d_t^{*2}(\theta)g(d_t^*(\theta))d_t(\theta) + 2d_t(\theta)G(d_t(\theta))$$
$$- d_t^2(\theta)g(d_t(\theta)) - 2\{K(d_t(\theta)) - K(-\infty)\}] + 2\tau[d_t(\theta) - E(\varepsilon_t)]\},$$

$$\frac{\partial}{\partial B} E[Q_n(\theta;\tau)] = \frac{1}{n} \sum_{t=1}^{n} \sin \omega t \{(1-2\tau)[d_t^{*2}(\theta)g(d_t^*(\theta))d_t(\theta) + 2d_t(\theta)G(d_t(\theta))$$
$$- d_t^2(\theta)g(d_t(\theta)) - 2\{K(d_t(\theta)) - K(-\infty)\}] + 2\tau[d_t(\theta) - E(\varepsilon_t)]\},$$

$$\frac{\partial}{\partial \omega} E[Q_n(\theta;\tau)] = \frac{1}{n} \sum_{t=1}^{n} (-At\sin \omega t + Bt\cos \omega t)\{(1-2\tau)[d_t^{*2}(\theta)g(d_t^*(\theta))d_t(\theta)$$
$$+ 2d_t(\theta)G(d_t(\theta)) - d_t^2(\theta)g(d_t(\theta))$$
$$- 2\{K(d_t(\theta)) - K(-\infty)\}] + 2\tau[d_t(\theta) - E(\varepsilon_t)]\}.$$

Now, note that

$$\frac{\partial}{\partial A} E[Q_n(\theta;\tau)] = \frac{1}{n} \sum_{t=1}^{n} \frac{\partial}{\partial A} E(X_t) = \frac{1}{n} \sum_{t=1}^{n} [\frac{\partial}{\partial A} X_t]$$

and

$$\frac{\partial}{\partial A} X_t = \begin{cases} 2\tau\cos \omega t[d_t(\theta) - \varepsilon_t], & \text{if } \varepsilon_t > d_t(\theta), \\ 2(1-\tau)\cos \omega t[d_t(\theta) - \varepsilon_t], & \text{if } \varepsilon_t < d_t(\theta). \end{cases}$$

Then,

$$E[\frac{\partial}{\partial A} X_t] = 2\cos \omega t[(1-\tau)\int_{-\infty}^{d_t(\theta)} d_t(\theta) dG(\varepsilon_t) + \tau \int_{d_t(\theta)}^{\infty} d_t(\theta) dG(\varepsilon_t)$$
$$- \{(1-\tau)\int_{-\infty}^{d_t(\theta)} \varepsilon_t dG(\varepsilon_t) + \tau \int_{d_t(\theta)}^{\infty} \varepsilon_t dG(\varepsilon_t)\}].$$

Since $d_t(\theta) = f_t(\theta) - f_t(\theta_0) = 0$ at $\theta_0 = (A_0, B_0, \omega_0)$, and by the assumption B3,

$$E[\frac{\partial}{\partial A} X_t]\Big|_{\theta=\theta_0} = -2\cos \omega_0 t[(1-\tau)\int_{-\infty}^{0} \varepsilon_t dG(\varepsilon_t) + \tau \int_{0}^{\infty} \varepsilon_t dG(\varepsilon_t)] = 0.$$

Then we obtain the result

$$\frac{\partial}{\partial \theta} Q(\theta;\tau)\Big|_{\theta=\theta_0} = \lim_{n\to\infty} \frac{\partial}{\partial \theta} E[Q_n(\theta;\tau)]\Big|_{\theta=\theta_0} = (0,0,0).$$

Now, if we define $[\alpha(n)]_{3\times 3}$ and $P[d_t(\theta)]$ as follows

$$[\alpha(n)]_{3\times 3} = \frac{\partial^2 E[Q_n(\theta;\tau)]}{\partial \theta^t \partial \theta}\Bigg|_{\theta=\theta_0}$$

and

$$P[d_t(\theta)] = 2G[d_t(\theta)] - 2d_t(\theta)G[d_t(\theta)] - d_t^2(\theta)\frac{\partial}{\partial d_t(\theta)} G[d_t(\theta)].$$

Then, we know that :

$$\frac{\partial^2 E[Q_n]}{\partial A \partial A} = \frac{1}{n} \sum_{t=1}^{n} \cos^2 \omega t \{(1-2\tau)P[d_t(\theta)] + 2\tau\},$$
$$\frac{\partial^2 E[Q_n]}{\partial B \partial A} = \frac{1}{2n} \sum_{t=1}^{n} \sin 2\omega t \{(1-2\tau)P[d_t(\theta)] + 2\tau\}$$

$$\frac{\partial^2 E[Q_n]}{\partial\omega\partial A} = \frac{1}{n}\left(-\frac{A}{2}\sum_{t=1}^{n} t\sin 2\omega t + B\sum_{t=1}^{n} t\cos^2\omega t\right)\{(1-2\tau)P[d_t(\theta)]+2\tau\},$$

$$\frac{\partial^2 E[Q_n]}{\partial B\partial B} = \frac{1}{n}\sum_{t=1}^{n}\sin^2\omega t\,\{(1-2\tau)P[d_t(\theta)]+2\tau\},$$

$$\frac{\partial^2 E[Q_n]}{\partial\omega\partial B} = \frac{1}{n}\left(-A\sum_{t=1}^{n} t\sin^2\omega t + \frac{B}{2}\sum_{t=1}^{n} t\sin 2\omega t\right)\{(1-2\tau)P[d_t(\theta)]+2\tau\},$$

$$\frac{\partial^2 E[Q_n]}{\partial\omega\partial\omega} = \frac{1}{n}\sum_{t=1}^{n}(-A t\sin\omega t + B t\cos\omega t)^2\,\{(1-2\tau)P[d_t(\theta)]+2\tau\}.$$

Since $d_t(\theta_0)=0$, $P(d_t(\theta_0))=P(0)=2G(0)+o(1)$. Let $S_0=(1-2\tau)2G(0)+2\tau$,
then we have following results when $n\to\infty$:

$$a(n)_{11} = \left.\frac{\partial^2 E[Q_n(\theta)]}{\partial A\partial A}\right|_{\theta=\theta_0} = \frac{1}{n}\sum_{t=1}^{n}\cos^2\omega_0 t\{(1-2\tau)P[d_t(\theta_0)]+2\tau\} = \frac{1}{2}S_0 + o(1)$$

$$a(n)_{22} = \left.\frac{\partial^2 E[Q_n(\theta)]}{\partial B\partial B}\right|_{\theta=\theta_0} = \frac{1}{n}\sum_{t=1}^{n}\sin^2\omega_0 t\{(1-2\tau)P[d_t(\theta_0)]+2\tau\} = \frac{1}{2}S_0 + o(1)$$

$$a(n)_{33} = \left.\frac{\partial^2 E[Q_n(\theta)]}{\partial\omega\partial\omega}\right|_{\theta=\theta_0} = \frac{1}{n}\sum_{t=1}^{n}(-A_0 t\sin\omega_0 t + B_0 t\cos\omega_0 t)^2 = S_0 + o(1).$$

Now, for the simplicity of $[a(n)]_{3\times3}$ consider the following facts:

$$\frac{1}{n}\sum_{t=1}^{n}\cos(2\omega_0 t) = \frac{1}{n}\sum_{t=1}^{n}\frac{e^{2\omega_0 ti} - e^{-2\omega_0 ti}}{2} = o(1).$$

Similarly, $\dfrac{1}{n}\displaystyle\sum_{t=1}^{n}\sin(2\omega_0 t) = o(1)$, and so we have

$$\frac{1}{n}\sum_{t=1}^{n}\cos^2\omega_0 t = \frac{1}{2}S_0 + o(1), \quad \frac{1}{n}\sum_{t=1}^{n}\sin^2\omega_0 t = \frac{1}{2}S_0 + o(1),$$

and,

$$\frac{1}{n}\sum_{t=1}^{n}(-A_0 t\sin\omega_0 t + B_0 t\cos\omega_0 t)^2$$

$$= A_0^2\frac{1}{n}\sum_{t=1}^{n}t^2\sin^2\omega_0 t - A_0 B_0\frac{1}{n}\sum_{t=1}^{n}t^2\sin 2\omega_0 t + B_0^2\frac{1}{n}\sum_{t=1}^{n}t^2\cos^2\omega_0 t.$$

So, $\dfrac{1}{n^2}a(n)_{3\times3} = \dfrac{A_0^2+B_0^2}{6}S_0 + o(1).$

Likewise the above processes, we obtain the results, when $n\to\infty$ :

$$a(n)_{12} = a(n)_{21} = \frac{1}{n}\sum_{t=1}^{n}\cos\omega_0 t\sin\omega_0 t S_0 = o(1),$$

$$a(n)_{13} = a(n)_{31} = \frac{1}{n}\left(-A_0\sum_{t=1}^{n} t\cos\omega_0 t\sin^2\omega_0 t + B_0\sum_{t=1}^{n} t\cos^2\omega_0 t\right)S_0.$$

Then, $\dfrac{1}{n}a(n)_{13} = \dfrac{1}{4}B_0 S_0 + o(1).$

$$a(n)_{23} = a(n)_{32} = \frac{1}{n}\left(-A_0\sum_{t=1}^{n} t\sin^2\omega_0 t + B_0\sum_{t=1}^{n} t\cos\omega_0 t\sin^2\omega_0 t\right)S_0.$$

Then, $\dfrac{1}{n} a(n)_{23} = \dfrac{1}{4} A_0 S_0 + o(1)$.

Hence, if we apply above results, we get

$$\lim_{n\to\infty}\left|\frac{\partial^2}{\partial\theta\partial\theta} E[\,Q_n(\theta,\tau)]\right|_{\theta=\theta_0} = \lim_{n\to\infty} n^2 \begin{vmatrix} \dfrac{1}{2} S_0 & 0 & -\dfrac{A_0 S_0}{4} \\ 0 & \dfrac{1}{2} S_0 & \dfrac{B_0 S_0}{4} \\ \dfrac{B_0 S_0}{4} & -\dfrac{A_0 S_0}{4} & \dfrac{(A_0^2 + B_0^2)S_0}{6} \end{vmatrix}$$

$$= \lim_{n\to\infty} n^2 \frac{S_0^3}{24}\left[(A_0 + \frac{3}{4} B_0)^2 + \frac{7}{16} B_0^2\right] > 0,$$

and we have the leading principal minors are positive. Thus $[a(n)]_{3\times3}$ is positive-definite. So, we can conclude that at least $\theta_0$ is a local minimizer of $Q(\theta,\tau)$. Next we will show that $\theta_0$ is the global minimizer of $Q(\theta,\tau)$. Since $Q(\theta,\tau) = \lim_{n\to\infty} \dfrac{1}{n} \sum_{t=1}^{n} E(X_t)$, and by the simple calculations of $X_t$ in (A.1), $E(X_t) > 0$, that implies $Q(\theta,\tau) > 0$, for $\theta \neq \theta_0$. And we also have $Q(\theta_0;\tau) = R_n(\theta_0;\tau) - R_n(\theta_0;\tau) = 0$. That is $Q(\theta,\tau) = 0$ at $\theta = \theta_0$ and $Q(\theta,\tau) > 0$ for all $\theta \neq \theta_0$. Thus $\theta_0$ is also the global minimizer of $Q(\theta,\tau)$.

# References

[1] Jennrich, R., Asymptotic properties of nonlinear least squares estimations, *Ann. Math. Stat.*, 40, (1969), 633-643.

[2] Koenker, R. and Bassett, G, Regression Quantiles, *Econometrica*, 73, (1978), 618-621.

[3] Kundu, D. Asymptotic Theory of Least Squares Estimator of a particular nonlinear regression model, *Statistics and Probability Letters*, 18, (1993), 13-17.

[4] Kundu, D. and Mitra, A., Asymptotic Theory of Least Squares Estimator of a nonlinear time series regression model, Communication in Statistics-Theory, 23(1), (1996), 133-141.

[5] Tae Soo Kim, Hae Kyoung Kim and Seung Hoe Choi, Asymptotic Properties of the LAD Estimators of a Nonlinear Time Series Regression Model, *Journal of the Korean Statistical Society*, 29, 2, (2000), 187-199.

[6] Walker, A. M., On the Estimation of a Harmonic Component in a Time Series with Stationary Independent Residuals, *Biometrica*, 58, (1971), 21-36.

[7] White, H., Nonlinear regression on cross-section data, *Econometrica*, 48, (1980), 721-746.

[8] Whitney K. Newey and James L. Powell, Asymmetric Least Squares Estimation and Testing, *Econometrica*, 4, (1987), 819-847.

[9] Wu, C. F., Asymptotic Theory of nonlinear least squares estimation, *The Annals of statistics*, 9, (1981), 501-513.