The Korean Communications
in Statistics Vol. 8, No. 3, 2001
pp. 615-623

A Functional Central Limit Theorem for the Multivariate Linear Process Generated by Negatively Associated Random Vectors

Tae-Sung Kim[1] and Hye-Young Seo[2]

Abstract

A functional central limit theorem is obtained for a stationary multivariate linear process of the form $X_t = \sum_{u=0}^{\infty} A_u Z_{t-u}$ where $\{Z_t\}$ is a sequence of strictly stationary m-dimensional negatively associated random vectors with $EZ_t = O$ and $E\|Z_t\|^2 < \infty$ and $\{A_u\}$ is a sequence of coefficient matrices with $\sum_{u=0}^{\infty} \|A_u\| < \infty$ and $\sum_{u=0}^{\infty} A_u \neq 0_{m\times m}$.

Keywords : Functional central limit theorem, m-dimensional linear process, negatively associated random vector, maximal inequality.

1. Introduction and main result

A finite family $\{Y_1, \cdots, Y_n\}$ of random variables is called associated if

$$\mathrm{Cov}(f(Y_1, \cdots, Y_n), g(Y_1, \cdots, Y_n)) \geq 0$$

for all real coordinatewise nondecreasing functions, f, g on R^n such that this covariance exists. It is called negatively associated if for any disjoint subset A of $\{1, \cdots, n\}$ and any real coordinatewise nondecreasing functions f on $R^{|A|}$ and g on $R^{|A^c|}$

$$\mathrm{Cov}(f(X_i, i \in A), g(X_j, j \in A^c)) \leq 0,$$

1) Professor Division of Mathematics and Informational Statistics, WonKwang University Iksan, Jeonbuk 570 - 749, Korea,
E-mail : starkim@wonkwang.ac.kr
2) Lecturer Division of Mathematics and Informational Statistics, WonKwang University Iksan, Jeonbuk 570-749, Korea,
E-mail : seofaith@hanmail.net

where $|A|$ is the cardinality of a set A and A^c is the complement of A.

An infinite family of random variables is associated (negatively associated) if every finite subfamily is associated (negatively associated). These concepts of dependence were introduced by Esary et al.(1967) and Joag-Dev and Proschan(1983). Basic properties of associated and negatively associated random variables may be found in Esary et al.(1967), Joag-Dev and Proschan(1983), and Newman(1984).

Let $\{X_t, t=0, \pm 1, \cdots\}$ be an m-dimensional linear process of the form

$$X_t = \sum_{u=0}^{\infty} A_u Z_{t-u} \tag{1}$$

defined on a probability space (Ω, F, P), where $\{Z_t\}$ is a sequence of stationary m-dimensional negatively associated random vectors with $EZ_t = O$, $E\|Z_t\|^2 < \infty$ and positive definite covariance matrix Γ : $m \times m$. Throughout we shall assume that

$$\sum_{u=0}^{\infty} \|A_u\| < \infty \text{ and } \sum_{u=0}^{\infty} A_u \neq \mathbf{0}_{m \times m}, \tag{2}$$

where for any $m \times m$, $m \geq 1$, matrix $A = (a_{ij})$, $\|A\| = \sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|$ and $\mathbf{0}_{m \times m}$ denotes the $m \times m$ zero matrix. Further, let

$$T = \left(\sum_{j=0}^{\infty} A_j \right) \Gamma \left(\sum_{j=0}^{\infty} A_j' \right),$$

where the prime denotes transpose, and the matrix $\Gamma = (\sigma_{kj})$ with

$$\sigma_{kj} = E(Z_{1k} Z_{1j}) + \sum_{t=2}^{\infty} (E(Z_{1k} Z_{tj}) + E(Z_{1j} Z_{tk})). \tag{3}$$

Let $S_n = \sum_{t=1}^{n} X_t$, $(n \geq 0)$ $(S_0 = O)$, and define, for $n \geq 1$, the stochastic process ξ_n by

$$\xi_n(u) = n^{-\frac{1}{2}} T^{-\frac{1}{2}} [S_r + (nu - r) X_{r+1}], \quad r \leq nu < r+1, \tag{4}$$

where $r = 0, 1, \cdots, n-1$.

Fakhre-Zakeri and Lee(1993) derived a functional central limit theorem for m-dimensional linear process X_t of the form (1), where $\{Z_t\}$ is an m-dimensional i.i.d. sequence of random vectors and Fakhre-Zakeri and Lee(2000) also obtained a functional central limit theorem for an m-dimensional martingale difference sequence $E(Z_t | F_{t-1}) = O$ a.s., where F_t is the sub σ-algebra generated by Z_u, $u \leq t$.

In this paper we define negatively associated random vectors and consider a functional central limit theorem for them and we also prove a functional central limit theorem for an m-dimensional linear process generated by m-dimensional negatively associated random vectors.

We close this section by introducing a basic definition and a main result.

Definition 1.1. The m-dimensional random vectors $\{Z_1, \cdots, Z_n\}$ are said to be negatively associated if for every nonempty subset of A of $\{1, \cdots, n\}$ and for every coordinatewise nondecreasing functions f, g such that $Ef^2(Z_t, t \in A) < \infty$, $Eg^2(Z_s, s \in A^c) < \infty$,

$$\mathrm{Cov}(f(Z_t, t \in A), g(Z_s, s \in A^c)) \leq 0.$$

Infinitely many random vectors are negatively associated, if any finite subset of them is a set of negatively associated random vectors.

Theorem 1.2. Let $\{Z_t, t = 1, 2, \cdots\}$ be a stationary negatively associated sequence of m-dimensional random vectors with $E(Z_t) = O, E\|Z_t\|^2 < \infty$ and positive definite covariance matrix Γ as in (3) and let ξ_n be as in (4). Assume that

$$E\|Z_1\|^2 + 2 \sum_{t=2}^{\infty} \sum_{i=1}^{m} |E(Z_{1i} Z_{ti})| = \sigma^2 < \infty, \tag{5}$$

$$\sum_{t=n+1}^{\infty} |E(Z_{1i} Z_{ti})| = O(n^{-\rho}) \text{ for some } \rho > 0, \tag{6}$$

and

$$E\|Z_t\|^s < \infty \text{ for some } s > 2. \tag{7}$$

Then, as $n \to \infty$,

$$\xi_n \Rightarrow W^m,$$

where $\Rightarrow$ indicates weak convergence and W^m denotes an m-dimensional Wiener process on $C^m[0, 1]$, the space of all continuous functions f defined on [0, 1] into $\boldsymbol{R}^m$ equipped with the norm $\|f\|_\infty = \max_{1 \leq i \leq m} \sup_{0 \leq t \leq 1} |f_i(t)|$.

2. Proofs

It is not hard to carry out that the moment bounds for associated sequences, given in Birkel(1988), still hold for negatively associated sequences. In particular we have :

Lemma 2.1. Let $\{Y_i, i = 1, 2, \cdots\}$ be a stationary negatively associated sequence with $EY_i = 0$. Assume

$$\sum_{i=n+1}^{\infty} |E(Y_1, Y_i)| = O(n^{-\rho}) \quad \text{for some} \quad \rho > 0, \tag{8}$$

and

$$E|Y_i|^s < \infty \quad \text{for some} \quad s > 2. \tag{9}$$

Then there exist $r > 2$ and B not depending on n such that for all $n \geq 1$

$$E|S_n|^r \leq Bn^{\frac{r}{2}}, \tag{10}$$

where $S_n = Y_1 + \cdots + Y_n$.

Remark 1. Note that from Lemma 2.1 and Theorem 3.7.5 in Stout(1974) for $r > 2$

$$E(\max_{1 \le k \le n} |S_k|)^r = O(n^{\frac{r}{2}}) \tag{11}$$

follows under assumptions in Lemma 2.1.

Lemma 2.2. Let $\{Y_i, i = 1, 2, \cdots\}$ be a stationary sequence of negatively associated random variables with $EY_i = 0$, $EY_i^2 < \infty$. Assume that (8) and (9) hold.
Let $W_n(u) = (s\sqrt{n})^{-1} S_{[nu]}$, $0 \le u \le 1$, and $w(W_n, \delta) = \sup_{0 \le u \le \delta} \sup |W_n(u)|$ where $s^2 = EY_1^2 + \sum_{i=2}^{\infty} |\mathrm{Cov}(Y_1, Y_i)| < \infty$. Then

$$\lim_n \sup P\{w(W_n, \delta) > \varepsilon\} \to 0 \quad \text{as} \quad \delta \downarrow 0. \tag{12}$$

Proof. Let $\varepsilon > 0$ be given. Then

$$P\{w(W_n, \delta) > \varepsilon\} \le \sum_{i=0}^{[1/\delta]} P\{\max_{0 < k \le [n\delta]} |S_k| > n\varepsilon/3\}. \tag{13}$$

According to Lemma 2.1 we have for $r > 2$

$$E|S_n|^r = O(n^{\frac{r}{2}}). \tag{14}$$

Hence, Theorem 3.7.5 of Stout (1974) yields

$$E(\max_{1 \le k \le n} |S_k|)^r = O(n^{\frac{r}{2}}). \tag{15}$$

Using Markov's inequality, (12) and (14), we obtain

$$\begin{aligned} & P\{w(W_n, \delta) > \varepsilon\} \\ & \le \left(\frac{3}{\varepsilon}\right)^{-r} n^{-\frac{r}{2}} \sum_{i=0}^{[1/\delta]} E(\max_{0 < k \le [n\delta]} |S_k|)^r \\ & \le c(\varepsilon) n^{-\frac{r}{2}} \sum_{i=0}^{[1/\delta]} [n\delta]^{\frac{r}{2}} \\ & \le c(\varepsilon) n^{-\frac{r}{2}} ([1/\delta] + 1)(n\delta + 1)^{\frac{r}{2}} \\ & = c(\varepsilon)(\delta + 1/n)^{\frac{r}{2}} ([1/\delta] + 1). \end{aligned}$$

Hence,

$$\limsup_n P\{w(W_n, \delta) > \varepsilon\} \le c(\varepsilon)\delta^{\frac{r}{2}} ([1/\delta] + 1) \to 0, \quad \text{as } \delta \downarrow 0.$$

This proves (12) and completes the proof.

Remark 2. Note that $\{W_n(\cdot)\}$ satisfies the tightness by Lemma 2.2 and Theorem 15.5 of Billingsley(1968).

Lemma 2.3. Let $\{Z_t,\ t=1,2,\cdots\}$ be a strictly stationary negatively associated sequence of m-dimensional random vectors with $E(Z_t)=O$ and $E\|Z_t\|^2<\infty$. Define, for $u\in[0,1]$, $n\geq 1$,

$$\eta_n(u)=n^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}\Big(\sum_{t\leq r}Z_t+(nu-r)Z_{r+1}\Big),\quad r\leq nu<r+1, \tag{16}$$

where $r=0,1,\cdots,n-1$, and the covariance matrix $\Gamma=(\sigma_{kj})$ with

$\sigma_{kj}=E(Z_{1k}Z_{1j})+\sum_{t=2}^{\infty}[E(Z_{1k}Z_{tj})+E(Z_{1j}Z_{tk})]$. If (5), (6) and (7) hold then, as $n\to\infty$,

$$\eta_n\Rightarrow W^m.$$

Proof. To prove tightness of the sequence $\{\eta_n(\cdot),\ n\geq 1\}$ note that each of the coordinates $\{\eta_{ni};\ i=1,\cdots,m\}$ satisfies (11). Tightness of the coordinate sequences and thereby tightness of η_n itself follows by standard arguments from Lemma 2.2 and Theorem 15.5 of Billingsley(1968). Hence, it remains to show that the only possible limit points is Brownian motion in $\boldsymbol{R}^m$ with covariance structure $\Gamma=(\sigma_{kl})$. The proof is similar to that of Theorem 2 of Burton et al.(1986). For the sake of completeness we repeat it. Let $Y(\cdot)$ be a limit point $(\eta_n(\cdot))$. We have to prove

(i) $Y(u+h)-Y(u)$ has normal distribution with covariance $h\cdot\Gamma$;

(ii) Y has independent increments.

Let $a\in\boldsymbol{R}^m$ be a non-negative vectors. Then $\langle a,(Y(t+h)-Y(t))\rangle$ has a normal distribution with variance $a\Gamma a$ by the Newman result for negatively associated process(see Newman(1984)). Now we can apply Lemma of Burton(1986) which yields (i). Let $0\leq u_1<u_2<\cdots<u_k\leq 1$ be given.

$$(\eta_{n'}(u_1),\eta_{n'}(u_2)-\eta_{n'}(u_1),\cdots,\eta_{n'}(u_k)-\eta_{n'}(u_{k-1})),\quad (n')\subset\boldsymbol{N},$$

converges in distribution to

$$(Y(u_1),Y(u_2)-Y(u_1),\cdots,Y(u_k)-Y(u_{k-1})).$$

Let $a_1,\cdots,a_k\in\boldsymbol{R}^m$ be non-negative vectors. Then

$$(\langle a_1,\eta_{n'}(u_1)\rangle,\langle a_2,(\eta_{n'}(u_2)-\eta_{n'}(u_1))\rangle,\cdots,\langle a_k,(\eta_{n'}(u_k)-\eta_{n'}(u_{k-1}))\rangle)$$

converges in distribution to

$$(\langle a_1,Y(u_1)\rangle,\cdots,\langle a_k,(Y(u_k)-Y(u_{k-1}))\rangle).$$

The coordinates of the last vectors are hence negatively associated and by a simple computation involving (6) also uncorrelated, which together imply independence. Now we can

again apply Lemma of Burton(1986) to obtain the independence of the increments of the Y process. Thus the proof of Lemma 2.3 is complete.

Lemma 2.4. Let $\{Z_t,\ t = 1, 2, \cdots\}$ be a strictly stationary negatively associated sequence of m-dimensional random vectors with $E(Z_t) = O$ and $E\|Z_t\|^2 < \infty$. Let $\widetilde{X}_t = (\sum_{j=0}^{\infty} A_j) Z_t$, $\widetilde{S}_k = \sum_{t=1}^{k} \widetilde{X}_t$ and assume that (2), (5), (6) and (7) hold.

Then

$$n^{-\frac{1}{2}} \max_{1 \le k \le n} \|\widetilde{S}_k - S_k\| = o_p(1). \tag{17}$$

Proof. see Appendix.

Finally, we prove the main result by using Lemmas 2.3 and 2.4.

Proof of Theorem 1.2. Let $\widetilde{X}_t = \left(\sum_{j=0}^{\infty} A_j\right) Z_t$. Then we have

$E(\widetilde{X}_t) = 0$ and $E\|\widetilde{X}_t\|^2 < \infty$,

$$E\|\widetilde{X}_1\|^2 + 2\sum_{t=2}^{\infty}\sum_{i=1}^{m} |E(\widetilde{X}_{1i}\widetilde{X}_{ti})|$$

$$\le \left(\sum_{j=0}^{\infty} \|A_j\|\right)^2 [E\|Z_1\|^2 + 2\sum_{t=2}^{\infty}\sum_{i=1}^{m} |E(Z_{1i} Z_{ti})|] < \infty,$$

$$\sum_{t=n+1}^{\infty} |E(\widetilde{X}_{1i}\widetilde{X}_{ti})|$$

$$\le \left(\sum_{j=0}^{\infty} \|A_j\|\right)^2 \sum_{t=n+1}^{\infty} |E(Z_{1i} Z_{ti})| = O(n^{-\rho}) \text{ for some } \rho > 0,$$

$$E\|\widetilde{X}_t\|^s \le \left(\sum_{j=0}^{\infty} \|A\|\right)^s E\|Z_t\|^s < \infty \quad \text{for some } s > 2$$

by assumptions (2),(5), (6) and (7), respectively.

Thus $\{\widetilde{X}_t\}$ satisfies the conditions (5), (6) and (7). Further let $\widetilde{\xi}_n$ be the same as ξ_n defined in (4) with $\widetilde{S}_r$ and $\widetilde{X}_{r+1}$ in place of S_r and X_{r+1}, respectively, that is,

$$\widetilde{\xi}_n(u) = \left(\sum_{j=0}^{\infty} A_j\right) n^{-\frac{1}{2}} T^{-\frac{1}{2}} \left(\sum_{t \le r} Z_t + (nu - r) Z_{r+1}\right), \quad r \le nu < r+1,$$

where $r = 0, 1, \cdots, n-1$. Thus it follows from Lemma 2.3 that $\widetilde{\xi}_n \Rightarrow W^m$. Applying Lemma 2.4 and Theorem 4.1 of Billingsley(1968) we obtain that $\xi_n \Rightarrow W^m$, so the proof is complete.

Appendix

Proof of Lemma 2.4. First observe that

$$\begin{aligned}\tilde{S}_k &= \sum_{t=1}^{k}\Big(\sum_{j=0}^{k-t}A_j\Big)Z_t + \sum_{t=1}^{k}\Big(\sum_{j=k-t+1}^{\infty}A_j\Big)Z_t \\ &= \sum_{t=1}^{k}\Big(\sum_{j=0}^{t-1}A_jZ_{t-j}\Big) + \sum_{t=1}^{k}\Big(\sum_{j=k-t+1}^{\infty}A_j\Big)Z_t\end{aligned}$$

and thus,

$$\begin{aligned}\tilde{S}_k - S_k &= -\sum_{t=1}^{k}\Big\{\sum_{j=t}^{\infty}A_jZ_{t-j}\Big\} + \sum_{t=1}^{k}\Big(\sum_{j=k-t+1}^{\infty}A_j\Big)Z_t \\ &= I_1 + I_2 \text{ (say)}.\end{aligned}$$

To prove

$$n^{-\frac{1}{2}}\max_{1\le k\le n}|I_1| = o_p(1), \tag{A.1}$$

we observe that for $r > 2$

$$\begin{aligned}&n^{-\frac{r}{2}}E\max_{1\le k\le n}\Big\|\sum_{t=1}^{k}\sum_{j=t}^{\infty}A_jZ_{t-j}\Big\|^r \\ &= n^{-\frac{r}{2}}E\max_{1\le k\le n}\Big\|\sum_{j=1}^{\infty}\sum_{t=1}^{j\wedge k}A_jZ_{t-j}\Big\|^r \\ &\le n^{-\frac{r}{2}}\Big(\sum_{j=1}^{\infty}\|A_j\|\Big\{E\max_{1\le k\le n}\Big\|\sum_{t=1}^{j\wedge k}Z_{t-j}\Big\|^r\Big\}^{\frac{1}{r}}\Big)^r \\ &\le B\Big[\sum_{j=1}^{\infty}\|A_j\|\Big(\frac{j\wedge k}{n}\Big)^{\frac{1}{2}}\Big]^r.\end{aligned}$$

The first inequality above is obtained by Minkowski's inequality and the last inequality is obtained by (11). Finally, by the dominated convergence theorem the last term above tends to zero as $n \to \infty$. Thus (A.1) is proved by the Markov inequality. Next, we show that

$$n^{-\frac{1}{2}}\max_{1\le k\le n}\|I_2\| = o_p(1). \tag{A.2}$$

Write

$$I_2 = II_1 + II_2, \text{ where}$$

$$II_1 = A_1Z_k + A_2(Z_k + Z_{k-1}) + \cdots + A_k(Z_k + \cdots + Z_1)$$

and

$$II_2 = (A_{k+1} + A_{k+2} + \cdots)(Z_k + \cdots + Z_1).$$

Let p_n be a sequence of positive integers such that

$$p_n \to \infty \quad\text{and}\quad p_n/n \to 0 \quad\text{as}\quad n \to \infty. \tag{A.3}$$

Note that

$$
\begin{aligned}
& n^{-\frac{1}{2}} \max_{1 \le k \le n} \| II_2 \| \\
& \le \left(\sum_{i=0}^{\infty} \| A_i \| \right) n^{-\frac{1}{2}} \max_{1 \le k \le p_n} \| Z_1 + \cdots + Z_k \| \\
& \quad + \left(\sum_{i > p_n} \| A_i \| \right) n^{-\frac{1}{2}} \max_{1 \le k \le n} \| Z_1 + \cdots + Z_k \| \\
& = III + IV \quad (\text{say}).
\end{aligned}
$$

Note that for $r > 2$ and positive constants B_1, B_2

$$
\left(\sum_{i=0}^{\infty} \| A_i \| \right)^r n^{-\frac{r}{2}} E \max_{1 \le k \le p_n} \| Z_1 + \cdots + Z_k \|^r \le \left(\sum_{i=0}^{\infty} \| A_i \| \right)^r B_1 (p_n / n)^{\frac{r}{2}},
$$

$$
\left(\sum_{i > p_n} \| A_i \| \right)^r n^{-\frac{r}{2}} E \max_{1 \le k \le n} \| Z_1 + \cdots + Z_k \|^r \le \left(\sum_{i > p_n} \| A_i \| \right)^r B_2 .
$$

by (2), (11) and (A.3). Thus

$$
\begin{aligned}
III + IV &= O_p(p_n / n) + O_p\left(\sum_{i > p_n} \| A_i \| \right) \\
&= o_p(1).
\end{aligned}
$$

It remains to prove that

$$
Y_n = n^{-\frac{1}{2}} \max_{1 \le k \le n} \| II_1 \| = o_p(1).
$$

To this end, define for each $l \ge 1$

$$
II_{1,l} = B_1 Z_k + B_2 (Z_k + Z_{k-1}) + \cdots + B_k (Z_k + \cdots + Z_1),
$$

where

$$
B_k = \begin{cases} A_k, & k \le l \\ 0_{m \times m}, & k > l. \end{cases}
$$

Let $Y_{n,l} = n^{-\frac{1}{2}} \max_{1 \le k \le n} \| II_{1,l} \|$. Clearly, for each $l \ge 1$,

$$
Y_{n,l} = o_p(1). \tag{A.4}
$$

On the other hand,

$$
\begin{aligned}
Y_{n,l} - Y_n &\le n^{-\frac{1}{2}} \max_{1 \le k \le n} \left\| \sum_{i=1}^{k} (A_i - B_i)(Z_k + \cdots + Z_{k-i+1}) \right\| \\
&= \max_{l < k \le n} \left(\sum_{i=l+1}^{k} \| A_i \| \cdot \| Z_k + \cdots + Z_{k-i+1} \| \right) \\
&\le \left(\sum_{i > l} \| A_i \| \right) \max_{l < k \le n} \max_{l < i \le k} \| Z_k + \cdots + Z_{k-i+1} \| \\
&\le 2 \left(\sum_{i > l} \| A_i \| \right) \max_{l \le j \le n} \| Z_1 + \cdots + Z_j \| .
\end{aligned}
$$

From this result and (11), for any $\delta > 0$ and $r > 2$,

$$\lim_{l\to\infty}\lim_{n\to\infty}\sup P(|Y_{n,l} - Y_n| > \delta)$$

$$\leq \lim_{l\to\infty}\lim_{n\to\infty}\sup 2^r\delta^{-r}(\sum_{i>l}\|A_i\|)^r n^{-\frac{r}{2}} E\max_{1\leq j\leq n}\|Z_1+\cdots+Z_j\|^r \quad \text{(A.5)}$$

$$\leq C\delta^{-r}2^r\lim_{l\to\infty}(\sum_{i>l}\|A_i\|)^r = 0.$$

In view of (A.4) and (A.5), it follows from Theorem 4.2 of Billingsley (1968, p.25) that $Y_n = o_p(1)$. This completes the proof of Lemma 2.4.

Acknowledgement The first author thanks WonKwang University for providing the research assistant.

References

[1] Billingsley. P.(1968) *Convergence of Probability Measures.* Wiley, New York.

[2] Birkel, T.(1988) Moment bounds for associated sequences. *Annal of Probability,* Vol. 16 1184-1193.

[3] Birkel, P.(1993) A functional central limit theorem for positively dependent random variables, *Journal of Multivariate Analysis,* Vol. 44, 314-320.

[4] Burton, R. M., Dabrowski, A. D. and Dehling, H.(1986) An invariance principle for weakly associated random vectors. *Stochastic Processes and Applied,* Vol. 23, 301-306.

[5] Esary, J. Proschan, F. and Walkup, D.(1967) Association of random variables with applications. *Annals of Mathematical Statistics,* Vol. 38, 1466-1474.

[6] Fakhre-Zakeri, I. and Lee, S.(1993) Sequential estimation of the mean vector of a multivariate linear process. *Journal of Multivariate Analysis,* Vol. 47, 196-209.

[7]------ (2000) On a functional limit theorems for multivariate linear with applications to sequential estimation. *Journal of Statistics Plann & Inference,* Vol.83, 11-23.

[8] Joag-Dev, K. and Proschan, F.(1983) Negative association of random variables with applications. *Annals of Statistics,* Vol. 11, 286-295.

[9] Lehmann, E. L.(1966) Some concepts of dependence, *Annals of Mathematical Statistics,* Vol. 37, 1137-1153.

[10] Newman, C. M.(1984) Asymptotic independence and limit theorems for positively and negatively dependent random variables. *Inequalities in Statistics and Probab. IMS Lecture Notes Monograph Series.* Vol. 5, 127-140.

[11] Stout, W.(1974) *Almost sure convergence,* Academic Press, New York.