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PROJECTIVE SCHUR ALGEBRAS AS CLASS ALGEBRAS

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ABSTRACT. A projective Schur algebra associated with a partition of finite group G can be constructed explicitly by defining linear transformations of G. We will consider various linear transformations and count the number of equivalent classes in a finite group. Then we construct projective Schur algebra whose dimension is determined by the number of classes.

1. Introduction

Let G denote a finite group, F a field of characteristic $p > 0$ and $F^* = F \setminus \{0\}$ the multiplicative group of F with trivial G-action. For a 2-cocycle α in $Z^2(G, F^*)$, let $F^\alpha G$ be the twisted group algebra over F with F-basis $\{u_g | g \in G\}$, $u_1 = 1 = 1_{F^\alpha G}$ such that $u_g u_x = \alpha(g,x) u_{gx}$ for all $g, x \in G$.

Let $\mathcal{P} = \{\mathcal{E}_g | g \in G\}$ be a partition of G consisting of equivalence classes $\mathcal{E}_g$ of G. Denote by G_0 a set of representatives of the distinct $\mathcal{E}_g$ and by $o_g^+ = \sum_{x \in \mathcal{E}_g} u_x$ the class sum in $F^\alpha G$. If $\mathcal{E}_g^{-1} = \{x^{-1} | x \in \mathcal{E}_g\} \in \mathcal{P}$ for all $\mathcal{E}_g$, then the subalgebra $S = \oplus_{g \in G_0} F o_g^+$ of $F^\alpha G$ with unit element 1 is called a projective Schur algebra in $F^\alpha G$ with partition $\mathcal{P}$. If $\alpha = 1$, the subalgebra generated by $\sum_{x \in \mathcal{E}_g} x$ in FG is called a Schur algebra in FG associated with $\mathcal{P}$. We may refer to [1], [3], and [4].

In this paper we study projective Schur algebras with respect to certain partitions of G. Various equivalent classes of G afforded by some linear transformations of G will be studied and the number of equivalent classes will be calculated. Upon using the classes, we construct some projective Schur algebras where the dimension of the algebra is determined by the number of equivalent classes.

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Throughout the paper, we always let $H < G$ and use the same notation $\alpha \in Z^2(G, F^*)$ for the restriction of α to $H \times H$. For any G-action τ on X (X: any set), we denote by $\mathcal{O}_\tau(x)$ and $\mathrm{St}_\tau(x)$ the orbit and the stabilizer of $x \in X$ under τ.

2. Regular classes of finite groups

An element $g \in G$ is α-regular if $\alpha(g, x) = \alpha(x, g)$ for all x in the centralizer $C_G(g)$ of g. A conjugacy class containing an α-regular element g is called an α-regular class and denoted by $\mathcal{R}_G(g)$. A 2-cocycle α is standard if $\alpha(x, x^{-1}) = 1$ and $\alpha(x, g) = \alpha(x^{-1}gx, x)$ for all $x \in G$ and all α-regular $g \in G$. If F is an algebraically closed field of characteristic 0 then any cohomology class $\bar{\alpha} \in H^2(G, F^*)$ contains a standard cocycle. If α is standard then all the α-regular class sums in each regular class form an F-basis of the center $Z(F^\alpha G)$ of $F^\alpha G$, thus $\dim Z(F^\alpha G)$ is equal to the number of distinct α-regular classes of G.

For $H < G$, two elements g, x in G are said to be H-conjugate if $x = h^{-1}gh$ for some $h \in H$. Clearly H-conjugacy is an equivalent relation on G, thus we can say H-class in G and denote it by $\mathcal{C}_H(g) = \{h^{-1}gh \mid h \in H\}$. If $G = H$ then $\mathcal{C}_G(g)$ is the conjugate class of $g \in G$. An element $g \in G$ is called α-H-regular if $\alpha(g, h) = \alpha(h, g)$ for all $h \in C_H(g) = \{h \in H \mid h = g^{-1}hg\}$ the centralizer of g in H. An H-class with an α-H-regular element g is called an α-H-regular class of g, and denoted by $\mathcal{R}_H(g)$:

$$\mathcal{R}_H(g) = \{h^{-1}gh \mid h \in H,\ g : \alpha\text{-}H\text{-regular}\}.$$

If $\alpha = 1$ then $\mathcal{R}_H(g) = \mathcal{C}_H(g)$. Each conjugacy class of G is a union of H-classes in G and an H-class containing an element of H is a conjugacy class of H.

LEMMA 2-1. [5, (6.2.3)] *If F is an algebraically closed field of characteristic 0 (or a splitting field for $F^\alpha G$) and if α is standard, then the number of nonequivalent irreducible α-representations of G over F equals the dimension of $Z(F^\alpha G)$. Also α-H-regular class sums form an F-basis for the centralizer $C_{F^\alpha G}(F^\alpha H)$ in $F^\alpha G$.*

If $\alpha = 1$, α-regular class is the conjugacy class and the number of irreducible FG-module is that of classes of G which equals $\dim Z(FG)$. Thus $\dim Z(F^\alpha G) \leq \dim Z(FG) \leq \dim C_{FG}(FH)$ and $\dim Z(F^\alpha G) \leq \dim C_{F^\alpha G}(F^\alpha H) \leq \dim C_{FG}(FH)$.

Let F be any field of characteristic p and let E be an algebraic closure of F with Galois group $\mathcal{G} = \mathrm{Gal}(E/F)$. Then $E \otimes F^\alpha G = E^\alpha G$ is a twisted group algebra obtained from $F^\alpha G$ by extending the field of scalars to E.

Choose positive integers n and $m(\sigma)$ for each $\sigma \in \mathcal{G}$ satisfying

(1) $\exp(G)|n, \quad \text{and} \quad \varepsilon_{n_{p'}}^{\sigma} = \varepsilon_{n_{p'}}^{m(\sigma)} \quad \text{while} \quad m(\sigma) \equiv 1 \pmod{n_p}$

where n_p and $n_{p'}$ are p- and p'-parts of n such that $n = n_p n_{p'}$, and $\varepsilon_{n_{p'}}$ is a primitive $n_{p'}$-th root of unity in E. If $p = 0$ or $p \not| |G|$ then $n_p = 1$ and $n_{p'} = n_p$.

Two elements g, x in G are called F-conjugate if $x = z^{-1} g^{m(\sigma^{-1})} z$ for some $z \in G$ and $\sigma \in \mathcal{G}$. For $H < G$, since $\exp H$ divides $\exp G$, we may choose n and $m(\sigma)$ work for both G and H. If there is $(\sigma, h) \in \mathcal{G} \times H$ such that $x = h^{-1} g^{m(\sigma^{-1})} h$ then g and x are said to be F-H-conjugate. Since both F-conjugacy and F-H-conjugacy are equivalence relations, we have F-class ${}_F\mathcal{C}_G(g)$ and F-H-class ${}_F\mathcal{C}_H(g)$ in G. In fact,

$${}_F\mathcal{C}_H(g) = \{h^{-1} g^{m(\sigma^{-1})} h \mid h \in H,\ \sigma \in \mathcal{G}\}.$$

We first remark simple but useful properties of F-H-classes.

LEMMA 2-2. *For $H < G$, we have the following.*

(i) *Each F-class of G is a union of F-H-classes. An F-H-class containing an element of H is an F-class of H. Each F-classes of H are F-H-classes in G.*
(ii) *Each F-H-class, as a set, commutes with every element of H. The inverse of the elements of an F-H-class forms an F-H-class. The product of two class sums of F-H-classes is a class sum of F-H-class.*

Proof. Let $g \in G$. If an F-H-class ${}_F\mathcal{C}_H(g)$ contains an element $h \in H$ then $h = a^{-1} g^{m(\sigma^{-1})} a$ for some $a \in H$, $\sigma \in \mathcal{G}$. Since $g = (aha^{-1})^{m(\sigma)} = ah^{m(\sigma)} a^{-1} \in H$, we have ${}_F\mathcal{C}_H(g) = {}_F\mathcal{C}_H(k)$ for some $k \in H$, i.e., an F-class of H.

For any $a^{-1}\, g^{m(\sigma^{-1})}\, a \in {}_F\mathcal{C}_H(g)$ with $a \in H$, $\sigma \in \mathcal{G}$, we have $h \cdot a^{-1} g^{m(\sigma^{-1})} a = h a^{-1} g^{m(\sigma^{-1})} a h^{-1} \cdot h$ for all $h \in H$. This shows that $h \cdot {}_F\mathcal{C}_H(g) = {}_F\mathcal{C}_H(g) \cdot h$.

Finally, for $g_i \in G$ if $o_{g_i}^+ = \sum_{z_i \in {}_F\mathcal{C}_H(g_i)} z$ then $o_{g_1}^+ o_{g_2}^+ = \sum_{g_3 \in G} n_{g_3} o_{g_3}^+$ where n_{g_3} are nonnegative integers. This completes the proof. $\square$

For each $g \in G$, choose an element $v(g) \in E$ as an n-th root of $t(g)$ such that

$$(2) \qquad v(g)^n = t(g) \text{ where } t(g) = \prod_{i=1}^{n-1} \alpha(g^i, g) \in F.$$

Write $F^\alpha G = \Gamma$. Define maps $K_\Gamma : G \to \mathrm{Aut}(E^\alpha G)$ and $S_\Gamma : \mathcal{G} \to \mathrm{Aut}(E^\alpha G)$ by

$$K_\Gamma(x)u_g = u_x^{-1}u_g u_x \text{ and } S_\Gamma(\sigma)u_g = v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} u_g^{m(\sigma^{-1})}.$$

We also define $k_G : G \to \mathrm{Aut}(G)$ and $s_G : \mathcal{G} \to \mathrm{Aut}(G)$ by $k_G(x)g = x^{-1}gx$ and $s_G(\sigma)g = g^{m(\sigma^{-1})}$. Let $d_G = s_G \times k_G$ and $D_\Gamma = S_\Gamma \times K_\Gamma$. Then

$$d_G(\sigma, x)(g) = x^{-1} g^{m(\sigma^{-1})} x$$

and

$$D_\Gamma(\sigma, x)u_g = v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} u_x^{-1} u_g^{m(\sigma^{-1})} u_x.$$

It was proved in [6] that s_G and S_Γ are $\mathcal{G}$-actions on G and $E^\alpha G$ respectively, while d_G and D_Γ are $\mathcal{G} \times G$-actions on G and $E^\alpha G$ respectively. The mappings do not depend on the choices of $n, m(\sigma)$ and $v(g)$ in (1), (2).

For $\alpha \in Z^2(G, F^*)$, an element $g \in G$ is called (F, α)-regular if

$$v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} u_x^{-1} u_g^{m(\sigma^{-1})} u_x = u_g \quad \text{for all } (\sigma, x) \in \mathcal{G} \times G$$

such that $x^{-1} g^{m(\sigma^{-1})} x = g$. An F-conjugacy class is called an (F, α)-regular class ${}_F\mathcal{R}_G(g)$ if it contains an (F, α)-regular element g. Similarly, $g \in G$ is (F, α)-H-regular if $v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} u_h^{-1} u_g^{m(\sigma^{-1})} u_h = u_g$ for all $(\sigma, h) \in \mathcal{G} \times H$ such that $h^{-1} g^{m(\sigma^{-1})} h = g$. Thus an F-H-class containing an (F, α)-H-regular element is called an (F, α)-H-regular class and is denoted by

$${}_F\mathcal{R}_H(g) = \{h^{-1} g^{m(\sigma^{-1})} h \mid (\sigma, h) \in \mathcal{G} \times H,\ g : (F, \alpha)\text{-regular}\}.$$

For $g \in G$, $h \in H$ and $a \in E^\alpha G$, we denote the restrictions to $\mathcal{G} \times H$ as following:

$$D_\Gamma|_H : \mathcal{G} \times H \to \mathrm{Aut}(E^\alpha G)$$
$$\text{by } D_\Gamma|_H(\sigma, h)a = v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} u_h a^{m(\sigma^{-1})} u_h$$
$$d_G|_H : \mathcal{G} \times H \to \mathrm{Aut}(G)$$
$$\text{by } d_G|_H(\sigma, h)g = h^{-1} g^{m(\sigma^{-1})} h.$$

And $k_G|_H(h)g = h^{-1}gh$ while $K_\Gamma|_H(h)a = u_h^{-1} a u_h$.

We develop a convenient way to express regularity properties by the mappings d_G and D_Γ and by their stabilizers $\mathrm{St}_{k_G|_H}(g)$, $\mathrm{St}_{d_G|_H}(g)$, $\mathrm{St}_{K_\Gamma|_H}(a)$ and $\mathrm{St}_{D_\Gamma|_H}(a)$.

THEOREM 2-3. *An (F,α)-regular class is a union of F-classes which is also a union of F-H-classes of G. Moreover*

(i) *g is α-regular if and only if $St_{K_\Gamma}(u_g) = St_{k_G}(g)$, while g is (F,α)-regular if and only if $St_{D_\Gamma}(u_g) = St_{d_G}(g)$.*
(ii) *The α-regular class is $\{k_G(x)g \mid x \in G,\ St_{k_G}(g) = St_{K_\Gamma}(u_g)\}$, while (F,α)-regular class is $\{d_G(\sigma,x)g \mid (\sigma,x) \in \mathcal{G}\times G,\ St_{d_G}(g) = St_{D_\Gamma}(u_g)\}$.*
(iii) *g is α-H-regular if and only if $St_{K_\Gamma|_H}(u_g) = St_{k_G|_H}(g)$, while g is (F,α)-H-regular if and only if $St_{D_\Gamma|_H}(u_g) = St_{d_G|_H}(g)$.*

Proof. Similar to Lemma 2-2, it is easy to see that set of (F,α)-regular elements is a union of F-classes and an (F,α)-regular class is a union of (F,α)-H-regular classes. Due to the constructions of k_G and d_G, we have the following sets:
$\mathrm{St}_{k_G}(g) = \{x \in G \mid g = x^{-1}gx\}$, $\mathrm{St}_{K_\Gamma}(u_g) = \{x \in G \mid u_g = u_x^{-1}u_g u_x\}$,
$\mathrm{St}_{d_G}(g) = \{(\sigma,x) \in \mathcal{G}\times G \mid g = x^{-1}g^{m(\sigma^{-1})}x\}$
$\mathrm{St}_{D_\Gamma}(u_g) = \{(\sigma,x) \in \mathcal{G}\times G \mid v(g)^{\sigma^{-1}}v(g)^{-m(\sigma^{-1})}u_x^{-1}u_g^{m(\sigma^{-1})}u_x = u_g\}$.

For (i) and (ii), we may refer to [2]. Now for (iii), since $\mathrm{St}_{K_\Gamma|_H}(u_g) = \{h \in H \mid u_h^{-1}u_g u_h = u_g\}$ and $\mathrm{St}_{k_G|_H}(g) = \{h \in H \mid h^{-1}gh = g\}$ for any $g \in G$, it is easy to see that

$$\mathrm{St}_{K_\Gamma|_H}(u_g) \subseteq \mathrm{St}_{k_G|_H}(g) \quad \text{and} \quad \mathrm{St}_{D_\Gamma|_H}(u_g) \subseteq \mathrm{St}_{d_G|_H}(g).$$

If $g \in G$ is α-H-regular then $u_h^{-1}u_g u_h = u_g$ for any $h \in H$ such that $h^{-1}gh = g$, thus it is equivalent to say $\mathrm{St}_{K_\Gamma|_H}(u_g) \supseteq \mathrm{St}_{k_G|_H}(g)$, i.e., $\mathrm{St}_{K_\Gamma|_H}(u_g) = \mathrm{St}_{k_G|_H}(g)$.

On the other hand, we have

$$\mathrm{St}_{d_G|_H}(g) = \{(\sigma,h) \in \mathcal{G}\times H \mid h^{-1}g^{m(\sigma^{-1})}h = g\} \quad \text{and}$$

$\mathrm{St}_{D_\Gamma|_H}(u_g) = \{(\sigma,h) \in \mathcal{G}\times H \mid v(g)^{\sigma^{-1}}v(g)^{-m(\sigma^{-1})}u_h^{-1}u_g^{m(\sigma^{-1})}u_h = u_g\}$.
Therefore, g is (F,α)-H-regular if and only if $D_\Gamma|_H(\sigma,h)u_g = u_g$ for $(\sigma,h) \in \mathcal{G}\times H$ with $d_G|_H(\sigma,h)g = g$ if and only if $\mathrm{St}_{D_\Gamma|_H}(u_g) = \mathrm{St}_{d_G|_H}(g)$, this completes the proof. □

Similar to the notation $\mathrm{St}_{k_G}(g) = C_G(g)$ the centralizer, we let $\mathrm{St}_{d_G}(g) = {}_F C_G(g)$ be the F-centralizer. We also denote by $\mathbf{R}C_G(g) = \{x \in$

$G\ |x^{-1}gx = g, g : \alpha$-regular$\}$ the α-regular centralizer of g. Then we have

$$\begin{aligned}\mathbf{R}C_G(g) &= \{x \in G | gx = xg, \alpha(g,x) = \alpha(x,g)\} \\ &= \{x \in G | u_g u_x = u_x u_g\} = \{x \in G | K_\Gamma(x) u_g u_g\} = \mathrm{St}_{K_\Gamma}(u_g).\end{aligned}$$

We denote by

$${}_F\mathbf{R}C_G(g) = \{(\sigma, x) \in \mathcal{G} \times G | x^{-1} g^{m(\sigma^{-1})} x = g, g : (F,\alpha)\text{-regular}\}$$

the (F,α)-regular centralizer of g, then it is equal to $\mathrm{St}_{D_\Gamma}(u_g)$. Therefore the next corollary follows immediately from Theorem 2-3.

COROLLARY 2-4. *An element $g \in G$ is α-regular if and only if $C_G(g) = \mathbf{R}C_G(g)$, and g is (F,α)-regular if and only if ${}_FC_G(g) = {}_F\mathbf{R}C_G(g)$.*

We now count the number of F-classes of G that will extends parts of [7].

THEOREM 2-5. *Let $f(G)$ and $f(H)$ be the numbers of F-classes of G and H respectively, and $f^H(G)$ be the number of F-H-classes of G. Then*

(i) $f(G) = \frac{1}{|\mathcal{G}\times G|}\sum_{g\in G} |{}_FC_G(g)|$ *and* $f(H) = \frac{1}{|\mathcal{G}\times H|}\sum_{h\in H} |{}_FC_H(h)|$.

(ii) $f^H(G) = \frac{1}{|\mathcal{G}\times H|}\sum_{g\in G} |{}_FC_H(g)| = \frac{1}{|\mathcal{G}\times H|}\sum_{h\in H} |{}_FC_G(h)|$.

Proof. We first observe a general result about permutation representations. Let Ω be any group (possibly infinite) that act on a finite set X under the action $\tau : \Omega \to \mathrm{Perm}(X)$. Then the number of Ω-orbits in X, say $n_\Omega(X)$ is

$$n_\Omega(X) = \sum_{x\in X} \frac{1}{|\Omega : \mathrm{St}_\tau(x)|}$$

where $\mathrm{St}_\tau(x)$ is the stabilizer of x in Ω. And $|\Omega : \mathrm{St}_\tau(x)| = |\mathcal{O}(x)|$ the Ω-orbit of x. In case Ω is finite, $n_\Omega(X)$ equals $\frac{1}{|\Omega|}\cdot\sum_{x\in X}|\mathrm{St}_\tau(x)|$.

For orbits of the mappings described in (3), we have $\mathcal{O}_{k_G|_H}(g) = \{h^{-1}gh | h \in H\}$ which is the H-conjugacy class $\mathcal{C}_H(g)$ in G, while $\mathcal{O}_{d_G|_H}(g) = \{h^{-1}g^{m(\sigma^{-1})}h | (\sigma,h) \in \mathcal{G}\times H\}$ which is the F-H-conjugacy class ${}_F\mathcal{C}_H(g)$ in G.

Thus $\mathcal{G}\times G$-orbit in G under d_G is an F-class in G so that the number of F-classes in G is $\sum_{g\in G}\frac{1}{|\mathcal{G}\times G : \mathrm{St}_{d_G}(g)|}$. Since $\mathrm{St}_{d_G}(g) = {}_FC_G(g)$, it follows that $f(G) = \frac{1}{|\mathcal{G}\times G|}\sum_{g\in G}|{}_F C_G(g)|$. Similarly, we can conclude (i) by showing

$$f(H) = \sum_{h\in H}\frac{1}{|\mathcal{G}\times H : \mathrm{St}_{d_H}(h)|} \quad \text{and } \mathrm{St}_{d_H}(h) = {}_F C_H(h).$$

Considering $d_G|_H : \mathcal{G}\times H \to \mathrm{Aut}(G)$, a $\mathcal{G}\times H$-orbit in G under $d_G|_H$ is an F-H-class in G and the number $f^H(G)$ of F-H-classes in G is

$$\sum_{g\in G}\frac{1}{|\mathcal{G}\times H : \mathrm{St}_{d_G|_H}(g)|} = \frac{1}{|\mathcal{G}\times H|}\sum_{g\in G}|{}_F C_H(g)|.$$

Due to the relation $\sum_{g\in G}|{}_F C_H(g)| = \sum_{h\in H}|{}_F C_G(h)|$, (ii) follows immediately. □

THEOREM 2-6. *Under all the same notations in Theorem 2-5, we have the following.*

(i) $f(H) \le |G : H| f(G)$. *And* $f(G) \le |G : H| f(H)$ *with equality if and only if* ${}_F C_G(g)\cdot(\mathcal{G}\times H) = \mathcal{G}\times G$.

(ii) $f(G) \le f^H(G)$ *and* $f(H) \le f^H(G)$. *Moreover for any subgroups* K *and* L *with* $L < H < K < G$, *we have* $f^K(G) \le f^H(G)$ *and* $f^L(H) < f^L(G)$.

Proof. Due to Theorem 2-5, we have

$$\begin{aligned} f(H) &= \frac{1}{|\mathcal{G}\times H|}\sum_{h\in H}|{}_F C_H(h)| \\ &\le \frac{1}{|\mathcal{G}\times H|}\sum_{h\in H}|{}_F C_G(h)| \\ &\le \frac{|\mathcal{G}\times G|}{|\mathcal{G}\times H|}\frac{1}{|\mathcal{G}\times G|}|{}_F C_G(g)| \\ &= |G : H| f(G). \end{aligned}$$

Since ${}_F C_G(g)\cap(\mathcal{G}\times H) = {}_F C_H(g)$ for all $g \in G$, we have $|\mathcal{G}\times H : {}_F C_H(g)| \le |\mathcal{G}\times G : {}_F C_G(g)|$, that is, $|{}_F C_G(g)| \le |G : H|\cdot|{}_F C_H(g)|$. And the equality hold if and only if ${}_F C_G(g)\cdot(\mathcal{G}\times H) = \mathcal{G}\times G$. Therefore, we have

$$\begin{aligned} f(G) &= \frac{1}{|\mathcal{G}\times G|}\sum_{g\in G}|{}_FC_G(g)| \\ &\leq \frac{1}{|\mathcal{G}\times H|}\sum_{g\in G}|{}_FC_H(g)| \\ &= \frac{1}{|\mathcal{G}\times H|}\sum_{h\in H}|{}_FC_G(h)| \\ &\leq \frac{1}{|\mathcal{G}\times H|}|G:H|\sum_{h\in H}|{}_FC_H(h)| \\ &= |G:H|f(H). \end{aligned}$$

Moreover it follows that

$$\begin{aligned} f(G) &= \frac{1}{|\mathcal{G}\times G|}\sum_{g\in G}|{}_FC_G(g)| \\ &\leq \frac{1}{|\mathcal{G}\times H|}\sum_{g\in G}|{}_FC_H(g)| \\ &= f^H(G), \end{aligned}$$

$$\begin{aligned} f^H(G) &< \frac{1}{|\mathcal{G}\times H|}\sum_{g\in G}|{}_FC_G(g)| \\ &= \frac{|G|}{|H|}\frac{1}{|\mathcal{G}\times G|}\sum_{g\in G}|{}_FC_G(g)| \\ &= |G:H|f(G). \end{aligned}$$

For any $H < K < G$ and for $g \in G$, since ${}_FC_K(g)\cap(\mathcal{G}\times H) = {}_FC_H(g)$, we have $f^K(G) \leq f^H(G)$. On the other hand, for $L < H < G$ it also follows that

$$f^L(H) = \frac{1}{|\mathcal{G}\times L|}\sum_{h\in H}|{}_FC_L(h)| \leq \frac{1}{|\mathcal{G}\times L|}\sum_{g\in G}|{}_FC_L(g)| = f^L(G). \quad \square$$

COROLLARY 2-7. *For $H < G$, let $n(G)$ and $n^H(G)$ be the numbers of classes and H-classes in G respectively. Let $g, x \in G$ and $h \in H$. Then*

(i) $n^H(G) = \frac{1}{|H|}\sum_{g\in G}|C_H(g)| = \frac{1}{|H|}\sum_{h\in H}|C_G(h)|$.

(ii) $n(G) \leq n^H(G)$ *and* $n(H) \leq n^H(G)$. *Moreover* $n^H(G) < |G : H|n(G)$.

(iii) *For subgroups* $L < H < K$ *of* G, *we have*

$$n^L(H) \leq n^L(G),\ n^L(H) < |H:L|n(H) \ \text{and} \ n^K(G) \leq n^H(G).$$

Thus $n(G) = n^G(G) \leq n^K(G) \leq n^H(G) \leq n^{\{1\}}(G) = |G|$.

Proof. The class and H-class in G are determined by mappings $k_G(x)g = x^{-1}gx$ and $k_{G|H}(h)g = h^{-1}gh$. Thus

$$n(G) = \frac{1}{|G|}\sum_{g\in G}|\mathrm{St}_{k_G}(g)| = \frac{1}{|G|}\sum_{g\in G}|C_G(g)|.$$

And the rest of the proof follows from Theorem 2-6. □

THEOREM 2-8. *Let* $G = G_1 \times G_2$ *and* $H = H_1 \times H_2$ *with* $H_i < G_i$. *Then we have* $f(G) = |\mathcal{G}|f(G_1)f(G_2)$, $n(G) = n(G_1)n(G_2)$ *and* $f^H(G) = |\mathcal{G}|f^{H_1}(G_1)f^{H_2}(G_2)$.

Proof. It is not hard to see that ${}_FC_G(g) \cong {}_FC_{G_1}(g_1) \times {}_FC_{G_2}(g_2)$ for $g \in G$ such that $g = (g_1, g_2) \in G_1 \times G_2$. Thus we have

$$\begin{aligned} f(G) &= \frac{1}{|\mathcal{G}||G|}\sum_{g\in G}|{}_FC_G(g)| \\ &= \frac{1}{|\mathcal{G}|}\frac{1}{|G_1||G_2|}\sum_{g_1\in G_1}\sum_{g_2\in G_2}|{}_FC_{G_1}(g_1)||{}_FC_{G_2}(g_2)| \\ &= |\mathcal{G}|f(G_1)f(G_2) \end{aligned}$$

and similarly, we have

$$\begin{aligned} f^H(G) &= \frac{1}{|\mathcal{G}|}\frac{1}{|H_1||H_2|}\sum_{g_i\in G_i}|{}_FC_{G_1}(g_1)||{}_FC_{G_2}(g_2)| \\ &= |\mathcal{G}|f^{H_1}(G_1)f^{H_2}(G_2). \end{aligned}$$

□

3. Schur and projective Schur algebras

Let S be a projective Schur algebra of $F^\alpha G$ with partition $\mathcal{P} = \{\mathcal{E}_g | g \in G\}$ of G. If $\mathcal{P}$ is afforded by an action τ on G then we shall call S a projective Schur algebra associated with τ. For $H < G$, let S' be a projective Schur algebra of $F^\alpha H$ associated with partition $\mathcal{P}' = \{\mathcal{E}'_h | h \in H\}$ of H. For each $h \in H$, if $\mathcal{E}'_h = \cup\mathcal{E}_g$ for some $g \in G$ then S' is called a projective Schur subalgebra of S in $F^\alpha G$.

In this section we shall construct projective Schur algebras explicitly according to the linear transformations described in previous section.

THEOREM 3-1. *Let F be a field of any char$F = p > 0$.*

(i) *There are projective Schur algebras A_1 in $F^\alpha G$ and A_2 in $F^\alpha H$ associated with d_G and d_H respectively. And there is a projective Schur algebra A_3 in $F^\alpha G$ associated with $d_G|_H$.*

(ii) *There are projective Schur algebras B_i ($i = 1,2,3$) with partitions of (F,α)-regular classes in G, (F,α)-regular classes in H and (F,α)-H-regular classes in G respectively. And B_i is a projective Schur subalgebra of A_i.*

Proof. Obviously $\mathcal{O}_{d_G}(g)$ is the F-conjugate class ${}_F\mathcal{C}_G(g)$ of $g \in G$, $\mathcal{O}_{d_G}(g^{-1}) = \mathcal{O}_{d_G}(g)^{-1}$ and $\mathcal{O}_{d_G}(e) = \{e\}$. By taking $o_g^+ = \sum_{z\in\mathcal{O}_{d_G}(g)} u_z$ in $F^\alpha G$, $A_1 = \oplus_g F o_g^+$ is a projective Schur algebra in $F^\alpha G$ where the sum runs over g in the set of all representatives of distinct orbits.

Similarly, the set of orbits $\mathcal{O}_{d_G|_H}(g) = {}_F\mathcal{C}_H(g)$ ($g \in G$) makes a partition of G and $\mathcal{O}_{d_G|_H}(g^{-1}) = \mathcal{O}_{d_G|_H}(g)^{-1}$. Hence the algebra $A_3 = \oplus_g F o_g^+$ generated by the class sums $o_g^+ = \sum_{z\in\mathcal{O}_{d_G|_H}(g)} u_z \in F^\alpha G$ is a projective Schur algebra in $F^\alpha G$.

Consider the set $\mathcal{P}_1$ of (F,α)-regular classes ${}_F\mathcal{R}_G(g)$ in G. Then the set of (F,α)-regular classes of G forms the partition $\mathcal{P}_1 = \{{}_F\mathcal{R}_G(g) |\ g \in G\}$ of G, and clearly if g is (F,α)-regular then so is g^{-1}. Moreover ${}_F\mathcal{R}_G(g^{-1}) = {}_F\mathcal{R}_G(g)^{-1}$ and ${}_F\mathcal{R}_G(e) = \{e\}$, thus by taking r_g^+ the (F,α)-regular class sum of ${}_F\mathcal{R}_G(g)$ in $F^\alpha G$, we have an F-algebra B_1 generated by all r_g^+ where g is an element in distinct (F,α)-regular classes. This is a projective Schur algebra in $F^\alpha G$ associated with $\mathcal{P}_1$. Due to Theorem 2-2, each (F,α)-regular class containing $g \in G$ is a union of F-classes so that B_1 is a projective Schur subalgebra of A_1.

Let $\mathcal{P}_2 = \{{}_F\mathcal{R}_H(h) |\ h \in H\}$ be the set of (F,α)-regular classes in H and $\mathcal{P}_3 = \{{}_F\mathcal{R}_H(g) |\ g \in G\}$ the set of (F,α)-H-regular classes in G. Then similar construction can be applied to B_2 and B_3, so that B_i is a projective Schur subalgebra of A_i. □

THEOREM 3-2. *Let F be a field of any char$F = p > 0$ and all the notations A_i and B_i ($i = 1,2,3$) are as in Theorem 3-1.*

(i) *$dimA_1 = \frac{1}{|\mathcal{G}\times G|}\sum_{g\in G} |{}_FC_G(g)|$, $dimA_2 = \frac{1}{|\mathcal{G}\times H|}\sum_{h\in H} |{}_FC_H(h)|$ and $dimA_3 = \frac{1}{|\mathcal{G}\times H|}\sum_{g\in G} |{}_FC_H(g)|$, where ${}_FC_G(g)$ is the F-centralizer of g in G, and etc.*

(ii) *$dimB_1 = \frac{1}{|\mathcal{G}\times G|}\sum_{g\in {}_FG_\alpha} |{}_FC_G(g)|$, $dimB_2 = \frac{1}{|\mathcal{G}\times H|}\sum_{h\in {}_FH_\alpha} |{}_FC_H(h)$ and $dimB_3 = \frac{1}{|\mathcal{G}\times H|}\sum_{g\in {}_FG_{\alpha H}} |{}_FC_H(g)|$ where ${}_FG_\alpha$ [resp. ${}_FG_{\alpha H}$] is the subset of all (F,α) [resp. (F,α)-H]-regular elements in G.*

Proof. The dimensions of each algebra in (i) Theorem 3-1 are determined by the numbers of corresponding classes, hence (i) follows from Theorem 2-5.

Now for (ii), if $g \in {}_FG_\alpha$ then its F-conjugation $x^{-1}g^{m(\sigma^{-1})}x$ for all $(\sigma, x) \in \mathcal{G} \times G$ is (F,α)-regular. Thus $\psi_\alpha : \mathcal{G} \times G \to \mathrm{Perm}({}_FG_\alpha)$ defined by

$$\psi_\alpha(\sigma, x)(g) = x^{-1}g^{m(\sigma^{-1})}x \quad \text{with } (\sigma, x) \in \mathcal{G} \times G$$

is a $\mathcal{G} \times G$-action on ${}_FG_\alpha$. Then the orbit

$$\mathcal{O}_{\psi_\alpha}(g) = \{x^{-1}g^{m(\sigma^{-1})}x \mid (\sigma, x) \in \mathcal{G} \times G\}$$

is the (F, α)-regular class and the stabilizer

$$\mathrm{St}_{\psi_\alpha}(g) = \{(\sigma, x) \in \mathcal{G} \times G \mid v(g)^{\sigma^{-1}}v(g)^{-m(\sigma^{-1})}u_x^{-1}u_g^{m(\sigma^{-1})}u_x = u_g\}$$

is the ${}_F\mathbf{R}C_G(g)$ (see Corollary 2-4). Since g is (F,α)-regular, it follows that $\mathrm{St}_{\psi_\alpha}(g) = {}_FC_G(g)$.

Similarly, we also have a $\mathcal{G} \times H$-action $\phi_\alpha : \mathcal{G} \times H \to \mathrm{Perm}({}_FG_{\alpha H})$ such that $\phi_\alpha(\sigma, h)(g) = h^{-1}g^{m(\sigma^{-1})}h$ for $(\sigma, h) \in \mathcal{G} \times H$ and $g \in {}_FG_{\alpha H}$. Then the orbit $\mathcal{O}_{\phi_\alpha}(g) = {}_F\mathcal{R}_H(g)$ while the stabilizer $\mathrm{St}_{\phi_\alpha}(g) = \{(\sigma, h) \in \mathcal{G} \times H \mid h^{-1}g^{m(\sigma^{-1})}h = g,\ g : (F, \alpha)\text{-}H\text{-regular}\}$ is the ${}_F\mathbf{R}C_H(g)$, and consequently this is equal to ${}_FC_H(g)$. Therefore (ii) follows from Theorem 2-5. □

COROLLARY 3-3. *Assume that F is an algebraically closed field of characteristic 0.*

(i) *There are projective Schur algebras A_i $(i = 1, 2, 3)$ associated with k_G, k_H and $k_G|_H$ respectively, and B_i $(i = 1, 2, 3)$ with partitions as α-regular class of G, α-regular class of H and α-regular H-class of G respectively. Each B_i is a projective Schur subalgebra of A_i.*

(ii) *$dimA_1 = \frac{1}{|G|}\sum_g |C_G(g)|$, $dimA_2 = \frac{1}{|H|}\sum_h |C_H(h)|$ and $dimA_3 = \frac{1}{|H|}\sum_g |C_H(g)|$. Moreover $dimB_1 = \frac{1}{|G|}\sum_{g\in G_\alpha} |C_G(g)|$, $dimB_2 = \frac{1}{|H|}\sum_{h\in H_\alpha} |C_H(h)|$ and $dimB_3 = \frac{1}{|H|}\sum_{g\in G_{\alpha H}} |C_H(g)|$, where G_α [resp. $G_{\alpha H}$] is the set of all α [resp. α-H]-regular elements of G.*

(iii) *In particular if $\alpha = 1$ then $A_1 = Z(FG)$, $A_2 = Z(FH)$ and $A_3 = C_{FG}(FH)$. And for any $\alpha \in Z^2(G, F^*)$, $B_1 = Z(F^\alpha G)$, $B_2 = Z(F^\alpha H)$ and $B_3 = C_{F^\alpha G}(F^\alpha H)$*

Proof. In case that F is an algebraically closed field, we may consider $d_G = k_G$. Obviously $\mathcal{O}_{k_G}(g)$ is the conjugacy class $\mathcal{C}_G(g)$ of $g \in G$, $\mathcal{O}_{k_G}(g^{-1}) = \mathcal{O}_{k_G}(g)^{-1}$ and $\mathcal{O}_{k_G}(e) = \{e\}$. Thus (i) and (ii) follow immediately.

The algebra in FG generated by class sums is $Z(FG)$ and the algebra in FG generated by H-class sums is $C_{FH}(FG)$. Similarly, the algebra in $F^\alpha G$ generated by α-regular class sums is $Z(F^\alpha G)$ and the algebra in $F^\alpha G$ generated by α-regular H-class sums is $C_{F^\alpha G}(F^\alpha H)$. Hence (iii) follows immediately. □

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