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HARMONIC BERGMAN SPACES OF THE HALF-SPACE AND THEIR SOME OPERATORS

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ABSTRACT. On the setting of the half-space of the Euclidean n-space, we consider harmonic Bergman spaces and we also study properties of the reproducing kernel. Using covering lemma, we find some equivalent quantities. We prove that if $\lim_{i\to\infty} \frac{\mu(K_r(z_i))}{V(K_r(z_i))} = 0$ then the inclusion function $I : b^p \to L^p(H_n, d\mu)$ is a compact operator. Moreover, we show that if f is a nonnegative continuous function in L^∞ and $\lim_{z\to\infty} f(z) = 0$, then T_f is compact if and only if $f \in C_0(H_n)$.

1. Introduction

Let H_n be the open subset of the Euclidean space $\mathbf{R}^n$ given by

$$H_n = \{(x, y) : y > 0\},$$

where we have written a typical point $z \in \mathbf{R}^n$ as $z = (x, y)$, with $x \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}^+$, dV will be the usual n-dimensional volume measure on H_n and $B(z, r)$ the Euclidean ball with center z and radius r. For $1 \le p < \infty$, let

$$b^p = \{f \in h(H_n) : \int_{H_n} |f|^p dV < \infty\},$$

where $h(H_n)$ is the set of all harmonic functions on H_n. Then the harmonic Bergman space b^p is a closed subspace of $L^p(H_n, dV)$ ([2], [3], [5]).

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If $p = 2$ then $L^2(H_n, dV)$ is a Hilbert space and hence there is an orthogonal projection Q from $L^2(H_n, dV)$ onto b^2. For each $z \in H_n$, we define $\Lambda_z : b^2 \longrightarrow \mathbf{C}$ by $\Lambda_z(f) = f(z)$ for all $f \in b^2$. Then $\Lambda_z \in (b^2)^*$. Thus there exists a unique function $R(z, \cdot) \in b^2$ such that $f(z) = \int_{H_n} f(w)R(z,w)dV(w)$ for all $f \in b^2$ and $Q(f(z)) = \int_{H_n} f(w)\ R(z,w)\ dV(w)$. By Theorem 8.22 in [2], for $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n) \in H_n$,

$$R(z,w) = \frac{4}{nV(B)} \frac{n(z_n + w_n)^2 - |z - \overline{w}|^2}{|z - \overline{w}|^{n+2}}$$

which is called the reproducing kernel for b^2, where $\overline{w} = (w_1, \cdots, w_{n-1}, -w_n)$. The purpose of this paper is to study these reproducing kernels and compactness characterization for Toeplitz operators with nonnegative continuous symbols on the harmonic Bergman space of the half-plane. In Section 2, we point out how harmonic reproducing kernels behave differently from one's on the unit disk. In Section 3, we establish some properties for $R(z, \cdot)$ and the inclusion operator $I : b^p \longrightarrow L^p(H_n, d\mu)$, where μ is a positive Borel measure on H_n. In the last section, we give a characterization of the compactness of Toeplitz operators with nonnegative bounded symbols.

Throughout this paper, the letters C and C_1 denote some constants and we use the symbol $\approx$ to indicate that the quotient of two quantities is bounded above and below by constants when the variables vary.

2. The reproducing kernel

LEMMA 2.1. *For any $z, w \in H_n$, there is a constant C such that*

$$|R(z,w)| \leq \frac{C}{|z - \overline{w}|^n}.$$

Proof. For any $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in H_n,

$$\begin{aligned} |R(z,w)| &= \left| \frac{4}{nV(B)} \frac{n(z_n + w_n)^2 - |z - \overline{w}|^2}{|z - \overline{w}|^{n+2}} \right| \\ &\leq \frac{4}{nV(B)} \frac{n|z - \overline{w}|^2 + |z - \overline{w}|^2}{|z - \overline{w}|^{n+2}} \\ &= \frac{4}{nV(B)} \frac{n+1}{|z - \overline{w}|^n}. \end{aligned}$$

This completes the proof. □

PROPOSITION 2.2. *For* $1 < q \leq \infty$ *and* $z \in H_n$, $R(z, \cdot) \in b^q$.

Proof. For $x, s \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}^+$,

$$P_{H_n}\big((x,y),s\big) = \frac{2}{nV(B)}\frac{y}{|(x,y)-s|^n}$$

is the Poisson kernel for H_n and hence

$$\int_{\mathbf{R}^{n-1}} \frac{2}{nV(B)}\frac{y}{\left(|x-s|^2+y^2\right)^{n/2}}ds = 1.$$

If $w = (s,t)$ and $z = (x,y)$ where $s, x \in \mathbf{R}^{n-1}$ and $t, y \in \mathbf{R}^+$ then

$$\begin{aligned}
&\int_{H_n} |R(z,w)|^q dV(w) \\
&\leq C^q \int_{H_n} \frac{1}{|z-\overline{w}|^{nq}} dV(w) \text{ by Lemma 2.1} \\
&= C^q \int_0^\infty \int_{\mathbf{R}^{n-1}} \frac{1}{\left(|x-s|^2+(y+t)^2\right)^{nq/2}} dsdt \\
&\leq C^q \int_0^\infty \int_{\mathbf{R}^{n-1}} \frac{1}{(y+t)^{n(q-1)+1}} \frac{(y+t)}{\left(|x-s|^2+(y+t)^2\right)^{n/2}} dsdt \\
&= C_1 \int_0^\infty \frac{1}{(y+t)^{n(q-1)+1}} dt \\
&= C_1 \int_y^\infty \frac{1}{t^{n(q-1)+1}} dt < \infty.
\end{aligned}$$

Since $R(z,\cdot)$ is harmonic, $R(z,\cdot)$ is in b^q. □

LEMMA 2.3. *For* $1 < p < \infty$, *there exists* C *such that*

$$\int_{H_n} \frac{w_n^{-1/p}}{|z-\overline{w}|^n} dV(w) = C z_n^{-1/p}$$

for all $z \in H_n$.

Proof. Fix $z = (x, y) \in H_n$. Letting $w = (s, t)$ where $s \in \mathbf{R^{n-1}}$ and $t \in \mathbf{R}^+$, we have

$$\begin{aligned}\int_{H_n} \frac{t^{-1/p}}{|z-\overline{w}|^n} dV(w) &= \int_0^\infty \int_{\mathbf{R}^{n-1}} \frac{t^{-1/p}}{|(x,y)-(s,-t)|^n} dsdt \\ &= \int_0^\infty \frac{t^{-1/p}}{y+t} \int_{\mathbf{R}^{n-1}} \frac{y+t}{|(x,y)-(s,-t)|^n} dsdt \\ &= \int_0^\infty \frac{t^{-1/p}}{y+t} \frac{nV(B)}{2} dt \\ &= \frac{nV(B)}{2} \int_0^\infty \frac{t^{-1/p}}{y+t} dt \\ &= y^{-1/p} \frac{nV(B)}{2} \int_0^\infty \frac{t^{-1/p}}{1+t} dt.\end{aligned}$$

Since $\int_0^\infty \frac{t^{-1/p}}{1+t} dt < \infty$, $\frac{nV(B)}{2} \int_0^\infty \frac{t^{-1/p}}{1+t} dt$ is constants and hence $\int_{H_n} \frac{t^{-1/p}}{|z-\overline{w}|^n} dV(w) = Cy^{-1/p}$ for some constants C. □

Suppose that $p \in (1, \infty)$ and $f \in L^p(H_n, dV)$. We note that $R(z, \cdot)$ is harmonic. By the Lebesgue dominated convergence theorem,

$$Q(f)(z) = \int_{H_n} f(w)R(z,w)dV(w)$$

is harmonic. Suppose $\frac{1}{p} + \frac{1}{q} = 1$. By Lemma 2.1,

$$\begin{aligned}|Q(f(z))| &= \Big| \int_{H_n} f(w)R(z,w)dV(w) \Big| \\ &\le C \int_{H_n} |f(w)| \frac{1}{|z-\overline{w}|^n} dV(w) \\ &= C \int_{H_n} |f(w)| \frac{w_n^{1/pq} w_n^{-1/pq}}{|z-\overline{w}|^{n/p} |z-\overline{w}|^{n/q}} dV(w).\end{aligned}$$

By the Hölder's inequality and Lemma 2.3,

$$\begin{aligned}
&\int_{H_n} \left|Q(f(z))\right|^p dV(z) \\
\le\ & \int_{H_n} C^p \left|\int_{H_n} |f(w)| \frac{1}{|z-\overline{w}|^{n/p}|z-\overline{w}|^{n/q}} dV(w)\right|^p dV(z) \\
\le\ & C^p \int_{H_n}\int_{H_n} |f(w)|^p \frac{w_n^{1/q}}{|z-\overline{w}|^n} dV(w) \Big(\int_{H_n} \frac{w_n^{-1/p}}{|z-\overline{w}|^n} dV(w)\Big)^{p/q} dV(z) \\
=\ & C^p C_1^{p/q} \|f\|_p^p.
\end{aligned}$$

Thus $Q\colon L^p(H_n, dV) \longrightarrow b^p$ is a bounded linear operator.

We want to find some equivalent quantities of the reproducing kernel. To do so, for any $r \in (0,1)$ and any $z \in H_n$, we define $K_r(z) = \{w \in H_n : |w - z| < rz_n\}$. Then we have the following lemma([4]).

LEMMA 2.4. *For $r \in (0, \frac{1}{3})$, there exists a sequence $\{z_i\}$ in H_n such that (1) $\cup K_r(z_i) = H_n$ and (2) there is $M \in \mathbf{N}$ such that for each $z \in H_n$, $\left|\{i\colon z \in K_{3r}(z_i)\}\right| \le M$.*

Proof. Let $w_m = (s_m, t_m)$ and $B_m = B(w_m, \frac{1}{5}t_m)$ where $s_m \in \mathbf{Q}^{n-1}$ and $t_m \in \mathbf{Q}^+$. Then $\cup B_m = H_n$. Put $D_1 = B_1$. For $n \ge 2$, we define $D_n = B_k$, where k is the first element of the $\{i : B_i \cap (\cup_{j=1}^{n-1} D_j) = \varnothing\}$ and let $z_m = (x_m, y_m)$ denote the center of D_m where $x_m \in \mathbf{R}^{n-1}$ and $y_m \in \mathbf{R}^+$. Take any $z \in H_n$. Then $z \in B_m$ for some m. If $B_m \cap D_l = \varnothing$ for $l \le m-1$ then $D_m = B_m$ and hence $z \in K_r(z_m)$. If $B_m \cap D_l \ne \varnothing$ for some $l \le m-1$ then $t_m - y_l \le |t_m - y_l| \le |w_m - z_l| < \frac{r}{5}t_m + \frac{r}{5}y_l$, i.e., $t_m < \frac{5+r}{5-r}y_l$. Thus $|z - z_l| \le |z - w_m| + |w_m - z_l| < \frac{r}{5}t_m + \frac{r}{5}t_m + \frac{r}{5}y_l < \frac{2r}{5}\frac{5+r}{5-r}y_l + \frac{r}{5}y_l = \frac{r(10+2r+5-r)}{5(5-r)}y_l < ry_l$. This implies $z \in K_r(z_l)$.

Take any $z = (x, y)$ in H_n. Let $N_z = \{m : |z - z_m| < 3ry_m\}$. For $m \in N_z$ and $w \in K_{\frac{r}{5}}(z_m)$, $|z - w| \le |z - z_m| + |z_m - w| < 3ry_m + \frac{r}{5}y_m = \frac{16r}{5}y_m < \frac{16r}{5(1-3r)}y$ and hence $K_{\frac{r}{5}}(z_m) \le K_{\frac{16r}{5(1-3r)}}(z)$. Since $\{K_{\frac{r}{5}}(z_m)\}$ is disjoint and

$$\begin{aligned}
\sum_{m\in N_z} |K_{\frac{r}{5}}(z_m)| &= C\pi \sum_{m\in N_z} \Big(\frac{r}{5}y_m\Big)^n \\
&> C\pi\Big(\frac{r}{5}\frac{y}{(1+3r)}\Big)^n |N_z|, \quad |N_z| < \Big(\frac{16(1+3r)}{1-3r}\Big)^n.
\end{aligned}$$

Thus $\{N_z : z \in H_n\}$ is uniformly bounded. □

PROPOSITION 2.5. *For $z \in H_n$ and $w \in K_r(z)$, $R(z, w) \approx \frac{1}{z_n^n}$.*

Proof. Since $\big|R(z,w)\big| \leq \dfrac{C}{|z-\overline{w}|^n}$ for all z,$w \in H_n$, $|R(z,w)| \leq \dfrac{C}{z_n^n}$. Since $n(z_n+w_n)^2 > |z-\overline{w}|^2$,

$$|R(z,w)| = \frac{4}{nV(B)}\left|\frac{n(z_n+w_n)^2-|z-\overline{w}|^2}{|z-\overline{w}|^{n+2}}\right| \geq \frac{C_1}{z_n^n}$$

for some C_1 and hence $R(z,w) \approx \dfrac{1}{z_n^n}$. □

PROPOSITION 2.6. *For $1 < p < \infty$ and $z \in H_n$,*

$$\|R(z,\cdot)\|_p \approx z_n^{-n(p-1)/p}.$$

Proof. By Proposition 2.5,

$$\begin{aligned}\|R(z,\cdot)\|_p^p &= \int_{H_n} \big|R(z,w)\big|^p dV(w)\\ &\geq \int_{K_r(z)} \big|R(z,w)\big|^p dV(w)\\ &= \int_{K_r(z)} \frac{C}{z_n^{np}} dV(w)\\ &= CC_1\frac{z_n^n}{z_n^{np}} = CC_1 z_n^{-n(p-1)}.\end{aligned}$$

Note that

$$\begin{aligned}\|R(z,\cdot)\|_p^p &= \int_{H_n} |R(z,w)|^p dV(w)\\ &\leq C_1 \int_{H_n} \frac{1}{|z-\overline{w}|^{np}} dV(w)\\ &\leq C_1C_2 \int_0^\infty \frac{1}{(z_n+w_n)^{n(p-1)+1}} dw_n\\ &= C_1C_2 \int_{z_n}^\infty \frac{1}{w_n^{n(p-1)-1}} dw_n\\ &= C_1C_2 z_n^{-n(p-1)}.\end{aligned}$$

The proof is complete. □

3. The embedding operator

Suppose that $1 \le p < \infty$, μ is a positive Borel measure on H_n and $\{K_r(z_i)\}$ is the sequence in Lemma 2.4. Let $I : b^p \longrightarrow L^p(H_n, d\mu)$ be the inclusion function. Suppose that $\dfrac{\mu(K_r(z_i))}{V(K_r(z_i))} < N$ for all $i = 1, 2, \cdots$ and M is the multiplicity in Lemma 2.4. Then we can show that I is a function. To do so, we need the following:

LEMMA 3.1. *For $0 < r < t < 1$ and $1 \le p < \infty$, there exists a finite constant C such that $|f(w)|^p \le \dfrac{C}{V(K_t(z))} \displaystyle\int_{K_t(z)} |f|^p dV$ for all $z \in H_n$, $w \in K_r(z)$ and all harmonic functions f on H_n.*

Proof. Let $z \in H_n$ and let $w \in K_r(z)$. Since $r < t$, for any harmonic function f on H_n,

$$\begin{aligned} |f(w)|^p &= \left|\frac{1}{V\big(B(w,(t-r)z_n)\big)} \int_{B\left(w,(t-r)z_n\right)} f dV\right| \\ &\le \frac{{C_1}^p}{V\big(K_t(z)\big)} \int_{K_t(z)} |f|^p dV \quad \text{for some constant } C_1. \end{aligned}$$

This implies the result. □

Take any u in b^p. Then

$$\begin{aligned} \int_{H_n} |u(z)|^p d\mu(z) &\le \sum_{i=1}^{\infty} \int_{K_r(z_i)} |u(z)|^p d\mu(z) \\ &\le \sum_{i=1}^{\infty} \mu\big(K_r(z_i)\big) \times \sup_{z \in K_r(z_i)} |u(z)|^p \\ &= C \sum_{i=1}^{\infty} \frac{\mu\big(K_r(z_i)\big)}{V\big(K_r(zi)\big)} \int_{K_r(z_i)} |u(z)|^p dV \\ &\le CN \sum_{i=1}^{\infty} \int_{K_{3r}(z_i)} |u(z)|^p dV(z) \\ &\le CNM \int_{H_n} |u(z)|^p dV(z). \end{aligned}$$

Since $u \in b^p$, $I : b^p \longrightarrow L^p(H_n, d\mu)$ is a function. In fact, we can show that I is compact whenever $\displaystyle\lim_{n\to\infty} \frac{\mu\big(K_r(z_i)\big)}{V\big(K_r(z_i)\big)} = 0$.

LEMMA 3.2. *For $1 < p < \infty$, $b^p \cap L^\infty$ is dense in b^p.*

Proof. Take any $\varepsilon > 0$ and any f in b^p. For each $\delta > 0$ and any $z = (x, y)$, let $f_\delta(z) = f(x, y+\delta)$. Then $f_\delta \in b^p$. Since $C_C(H_n)$ is dense in L^p, there is $g \in C_C(H_n)$ such that $\|g - f\|_p < \varepsilon$. Since $\lim_{\delta\to 0} \|g_\delta - g\|_p = 0$, there is $\delta_0 > 0$ such that for $0 < \delta < \delta_0$, $\|g_\delta - g\|_p < \varepsilon$ and hence $\|f_\delta - f\|_p \le \|f_\delta - g_\delta\|_p + \|g_\delta - g\|_p + \|g - f\|_p < 3\varepsilon$. Then for any $w = (s,t) \in H_n$,

$$\begin{aligned}
|f_\delta(w)|^p &= |f(s, t+\delta)|^p \\
&= \left| \frac{1}{V\big(B((s,t+\delta),\delta)\big)} \int_{B\left((s,t+\delta),\delta\right)} f(z)dV(z) \right|^p \\
&\le \frac{1}{V\big(B((s,t+\delta),\delta)\big)} \int_{H_n} |f(z)|^p dV(z).
\end{aligned}$$

This implies $f_\delta \in L^\infty$. Thus $b^p \cap L^\infty$ is dense in b^p. □

PROPOSITION 3.3. *For $1 < p < \infty$ and $z \in H_n$, $\dfrac{R(z,\cdot)}{\|R(z,\cdot)\|_p}$ converges weakly to 0 in b^p as $z_n \to 0$.*

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$. Take any v in $b^q \cap L^\infty$. Since $\|R(z,\cdot)\|_p \approx z_n^{-n(p-1)/p}$, $\left|\left\langle \dfrac{R(z,\cdot)}{\|R(z,\cdot)\|_p}, v \right\rangle\right| = \dfrac{1}{\|R(z,\cdot)\|_p} |v(z)| \approx z_n^{n/q} |v(z)|$ and hence $\dfrac{R(z,\cdot)}{\|R(z,\cdot)\|_p}$ converges weakly to 0 as $z_n \to 0$. □

LEMMA 3.4. *Let $1 < p < \infty$ and let $\{f_m\}$ be a sequence in b^p. Then $\{f_m\}$ converges weakly to f in b^p if and only if $\{\|f_m\|_p : m \in \mathbf{N}\}$ is bounded and $\{f_m\}$ converges uniformly to f on each compact subset of H_n.*

Proof. Suppose $\{f_m\}$ is a sequence in b^p such that $\{f_m\}$ converges weakly to f in b^p. For each $g \in b^p$, we define $\Lambda_g : \left(b^p\right)^* \longrightarrow \mathbf{C}$ by $\Lambda_g(v) = v(g)$ for all $v \in \left(b^p\right)^*$. Then $\|\Lambda_g\| = \|g\|_p$. We note that $\{f_m\}$ converges weakly to f if and only if $\lim_{m\to\infty} v(f_m) = v(f)$ for all $v \in \left(b^p\right)^*$ if and only if $\{\Lambda_{f_m}\}$ converges pointwise to f in $\left(b^p\right)^*$ and $\left(b^p\right)^*$ is a Banach space. By the uniform boundedness principle, $\sup\{\|\Lambda_{f_m}\| : m \in \mathbf{N}\} = \sup\{\|f_m\|_p : m \in \mathbf{N}\}$ is bounded. For any

$g \in b^p$, $z \in H_n$ and any compact subset K of H_n,

$$\begin{aligned} |g(z)|^p &= \left|\frac{1}{V(B(z,z_n)} \int_{B(z,z_n)} g(w)dV(w)\right|^p \\ &\leq \frac{1}{V(B(z,z_n))} \int_{B(z,z_n)} |g(w)|^p dV(w) \\ &\leq \frac{1}{V(B(z,z_n))} \|g\|_p^p. \end{aligned}$$

By the Arzela-Ascoli theorem, for any compact subset K, there is a subsequence $\{f_{m_k}\}$ of $\{f_m\}$ such that $\{f_{m_k}\}$ converges uniformly to f on K. Since $\{f_m\}$ converges pointwise to f and $\{f_{m_k}\}$ converges uniformly to f on K, $\{f_m\}$ converges uniformly to f on each compact subset on H_n.

Conversely, take any v in $(C_C(H_n))^*$. By the Riesz representation theorem, there exists a unique regular, complex-valued Borel measure μ such that $v(g) = \int_{H_n} g d\mu$ for all $g \in C_C(H_n)$. Note that $\{f_m\}$ converges pointwise to f. By the Lebesgue dominated convergence theorem, $\lim_{n\to\infty} v(f_m) = \lim_{m\to\infty} \int_{H_n} f_m d\mu = \int_{H_n} f d\mu = v(f)$ and hence $\{f_m\}$ converges weakly to f. □

THEOREM 3.5. *If* $\lim_{i\to\infty} \frac{\mu(K_r(z_i))}{V(K_r(z_i))} = 0$ *then the embedding operator* I *is compact.*

Proof. Suppose $\{f_m\}$ converges weakly to 0 in b^p. By Lemma 3.4, $\{f_m\}$ converges uniformly on each compact subset of H_n and $\{\|f_m\|_p : m \in \mathbf{N}\}$ is bounded. Let $\varepsilon > 0$ be given. Since $\lim_{i\to\infty} \frac{\mu(K_r(z_i))}{V(K_r(z_i))} = 0$, there is $k \in \mathbf{N}$ such that for $i \geq k$, $\frac{\mu(K_r(z_i))}{V(K_r(z_i))} < \varepsilon$. By Lemma 3.1, there is a constant C such that $|f_m(z)|^p \leq \frac{C}{V(K_{3r}(z_i))} \int_{K_{3r}(z_i)} |f_m|^p dV$

for all $z \in K_r(z_i)$. Then

$$\begin{aligned}
&\int_{H_n} |f_m(z)|^p d\mu(z) \\
&\leq \sum_{i=1}^{\infty} \int_{K_r(z_i)} |f_m(z_i)|^p d\mu(z) \\
&= \sum_{i=1}^{k} \int_{K_r(z_i)} |f_m(z)|^p d\mu(z) + \sum_{i=k+1}^{\infty} \int_{K_r(z_i)} |f_m(z)|^p d\mu(z) \\
&\leq \sum_{i=1}^{k} \int_{K_r(z_i)} |f_m(z)|^p d\mu(z) + C \sum_{i=k+1}^{\infty} \frac{\mu\big(K_r(z_i)\big)}{V\big(K_r(z_i)\big)} \int_{K_{3r}(z_i)} |f_m(z)|^p dV(z) \\
&\leq \sum_{i=1}^{k} \int_{K_r(z_i)} |f_m(z)|^p d\mu(z) + C\varepsilon M \int_{H_n} |f_m(z)|^p dV(z),
\end{aligned}$$

where M is the constant in Lemma 2.4.
Since $\{f_m\}$ converges uniformly to 0 each compact subset of H_n,

$$\lim_{m\to\infty} \sum_{i=1}^{k} \int_{K_r(z_i)} |f_m(z)|^p d\mu(z) = \sum_{i=1}^{k} \int_{K_r(z_i)} \lim_{m\to\infty} |f_m(z)|^p d\mu(z) = 0$$

and hence $\lim_{m\to\infty} \|I(f_k)\|_p^p = 0$. Thus I is compact. □

4. Toeplitz operators on harmonic Bergman spaces

We note that $Q : L^2(H_n, dV) \longrightarrow b^2$ is the Bergman projection. For $f \in L^\infty$, we define $T_f : b^2 \longrightarrow b^2$ by $T_f(g) = Q(fg)$ for all $g \in b^2$, which is called the Toeplitz operator with symbol f ([1]). Since $\|Q\| \leq 1$, $\|T_f\| \leq \|f\|_\infty$.

THEOREM 4.1. *Suppose $0 < r < 1$, $1 \leq p < \infty$ and μ is a positive Borel measure on H_n. Then*

$$\sup_{\substack{f\in b^p \\ f\neq 0}} \frac{\int_{H_n} |f|^p d\mu}{\int_{H_n} |f|^p dV} \approx \sup_{z\in H_n} \frac{\mu\big(K_r(z)\big)}{V\big(K_r(z)\big)} .$$

Proof. For $z \in H_n$, let $g(w) = R(z,w)^{2/p}$. Then $\int_{H_n} |g(w)|^p dV(w) = R(z,z)$. By Proposition 2.5,

$$\int_{H_n} |g(w)|^p d\mu(w) \geq \int_{K_r(z)} |R(z,w)|^2 d\mu(w) \approx \int_{K_r(z)} \frac{1}{z_n^{2n}} d\mu = \frac{1}{z_n^{2n}} \mu\big(K_r(z)\big)$$

Since $|R(z,w)| = C_1 \frac{1}{z_n^n}$ for some C_1,

$$\frac{\int_{H_n} |g(w)|^p d\mu(w)}{\int_{H_n} |g(w)|^p dV(w)} \geq C_2 \frac{\frac{1}{z_n^{2n}} \mu\big(K_r(z)\big)}{\frac{1}{z_n^n}} = C \frac{\mu\big(K_r(z)\big)}{V\big(K_r(z)\big)} \text{ for some } C_2 \text{ and } C.$$

This implies that $\sup\limits_{\substack{f \in b^p \\ f \not\equiv 0}} \dfrac{\int_{H_n} |f|^p d\mu}{\int_{H_n} |f|^p dV} \geq C \sup\limits_{z \in H_n} \dfrac{\mu\big(K_r(z)\big)}{V\big(K_r(z)\big)}$.

Suppose $\{K_r(z_i)\}$ is the sequence and M is the constant in Lemma 2.4. Let $f \in b^p$ be such that $f \not\equiv 0$. Then

$$\begin{aligned} \int_{H_n} |f|^p d\mu &\leq \sum_{i=1}^{\infty} \int_{K_r(z_i)} |f|^p d\mu \\ &\leq \sum_{i=1}^{\infty} \sup_{w \in K_r(z_i)} |f(w)|^p \mu\big(K_r(z_i)\big) \\ &\leq C \sum_{i=1}^{\infty} \frac{\mu\big(K_r(z_i)\big)}{V\big(K_{\frac{1+r}{2}}(z_i)\big)} \int_{K_{\frac{1+r}{2}}(z_i)} |f|^p dV \\ &\leq CM \sup_{z \in H_n} \frac{\mu\big(K_r(z)\big)}{V\big(K_r(z)\big)} \|f\|_p^p. \end{aligned}$$

Thus

$$\sup_{\substack{f \in b^p \\ f \not\equiv 0}} \frac{\int_{H_n} |f|^p d\mu}{\int_{H_n} |f|^p dV} \approx \sup_{z \in H_n} \frac{\mu\big(K_r(z)\big)}{V\big(K_r(z)\big)}. \qquad \square$$

PROPOSITION 4.2. *Let K be a compact subset of H_n. If f is in L^∞ and $f \equiv 0$ on $H_n \setminus K$ then T_f is compact.*

Proof. Let $\{g_m\}$ be a norm bounded sequence in b^2. Take any compact subset K_1 of H_n. By Hölder's inequality, for any $z \in K_1$, $|g_m(z)| \leq \int_{H_n} |g_m(w) R(z,w)| dV(w) \leq \|g_m\|_2 \|R(\cdot,w)\|_2$ and hence there is a harmonic function g on H_n and a subsequence $\{g_{m_k}\}$ of $\{g_m\}$ which converges uniformly on K_1 to g. Since T_f is continuous, $T_f(g_{m_k})$ converges to $T_f(g)$. Thus T_f is compact. $\square$

PROPOSITION 4.3. *Let f be a nonnegative function in L^∞. If there exists $r \in (0,1)$ such that $\lim_{z_n \to 0} \frac{1}{V(K_r(z))} \int_{K_r(z)} f(w)dV(w) = 0$, then T_f is compact.*

Proof. For each $k \in \mathbf{N}$, let $D_k = [-k,k] \times \cdots \times [-k,k] \times [\frac{1}{k}, k]$ and let $f_k(z) = f(z)\chi_{D_k}(z)$. Then D_k is compact. By Proposition 4.2, each T_{f_k} is compact. Then

$$\begin{aligned}
\|T_f - T_{f_k}\|^2 &= \sup_{\|u\|_2=1} \|(T_f - T_{f_k})(u)\|_2^2 \\
&= \sup_{\|u\|_2=1} \int_{H_n} \left|(fu - f_k u)(w) R(z,w)\right|^2 dV(w) \\
&\le \sup_{\|u\|_2=1} \int_{H_n \setminus D_k} \left|(fu - f_k u)(w) R(z,w)\right|^2 dV(w) \\
&\le C \sup_{\|u\|_2=1} \int_{H_n \setminus D_k} f^2 |u|^2 dV \text{ for some } C \\
&= C \sup_{\|u\|_2=1} \int_{H_n} \chi_{H_n \setminus D_k} f^2 |u|^2 dV \\
&\le C_1 C \sup_{z \in H} \frac{\int_{H_n} \chi_{K_r(z)} \chi_{H_n \setminus D_k} f^2 dV}{V(K_r(z))} \\
&\qquad \text{for some } C_1 \text{ by Theorem 4.1} \\
&\le C_1 C \|f\|_\infty \sup_{z \in H_n} \frac{\int_{H_n} \chi_{K_r(z) \setminus D_k} f^2 |u|^2 dV}{V(K_r(z))}.
\end{aligned}$$

By the assumption, $\lim_{k \to \infty} \|T_f - T_{f_k}\| = 0$. By Proposition 4.2, each T_{f_k} is compact and hence T_f also compact. □

THEOREM 4.4. *Let f be a nonnegative function in L^∞. Then the following are equivalent:*

(1) *T_f is compact.*

(2) *There exists $r \in (0,1)$ such that $\lim_{z_n \to 0} \frac{1}{V(K_r(z))} \int_{K_r(z)} f(w)dV(w) = 0$.*

(3) *For any $r \in (0,1)$, $\lim_{z_n \to 0} \frac{1}{V(K_r(z))} \int_{K_r(z)} f(w)dV(w) = 0$.*

Proof. It is clear that (3) implies (2). By Proposition 4.3, (2) implies (1). It is enough to show that (1) implies (3) to complete the proof.

Suppose T_f is a compact operator and $z = (x, y)$. For any $r \in (0, 1)$,

$$\begin{aligned}
&\frac{1}{V\big(K_r(z)\big)} \int_{K_r(z)} f(w)dV(w) \\
&\approx \frac{1}{z_n^n} \int_{K_r(z)} f(w)dV(w) \\
&\approx \int_{K_r(z)} f(w) \frac{|R(z,w)|^2}{\|R(z,\cdot)\|_2^2} dV(w) \text{ by Proposition 2.5 and Propositon 2.6} \\
&= \int_{K_r(z)} \frac{f(w)}{\|R(z,\cdot)\|_2} R(z,w) \int_{H_n} R(w,t)R(z,t)dV(t)dV(w) \\
&\le \int_{H_n} \frac{f(w)}{\|R(z,\cdot)\|_2^2} R(z,w) \int_{H_n} R(w,t)R(z,t)dV(t)dV(w) \\
&= \int_{H_n} \int_{H_n} \frac{f(w)}{\|R(z,\cdot)\|_2} R(z,w)R(t,w)dV(w) \frac{R(z,t)}{\|R(z,\cdot)\|_2} dV(t) \\
&= \int_{H_n} Q\Big(\frac{fR(z,\cdot)}{\|R(z,\cdot)\|_2}\Big)(t) \frac{R(z,t)}{\|R(z,\cdot)\|_2} dV(t) \\
&= \Big\langle T_f\Big(\frac{R(z,\cdot)}{\|R(z,\cdot)\|_2}\Big), \frac{R(z,\cdot)}{\|R(z,\cdot)\|_2}\Big\rangle.
\end{aligned}$$

By Proposition 3.3, $\lim\limits_{z_n \to 0} \frac{1}{V\big(K_r(z)\big)} \int_{K_r(z)} f(w)dV(w) = 0.$ □

THEOREM 4.5. *Suppose that f is a nonnegative continuous function in L^∞ and $\lim\limits_{z\to\infty} f(z) = 0$. Then the following are equivalent:*

(1) *T_f is compact.*
(2) *$\lim\limits_{z\to\partial H_n} f(z) = 0$.*
(3) *$f \in C_0(H_n)$.*

Proof. Suppose T_f is compact and $\lim\limits_{z\to z_0} f(z) > 0$ for some $z_0 \in \partial H_n$. Then there is $r > 0$ such that for $|z - z_0| < r$ and $z \in H_n$, $f(z) > \frac{A}{2}$, where $\lim_{z\to z_0} f(z) = A$. This contradicts the fact that for any $r \in (0,1)$, $\lim\limits_{z_n\to 0} \frac{1}{V\big(K_r(z)\big)} \int_{K_r(z)} f(w)dV(w) = 0.$

Conversely, take any $r \in (0, 1)$ any $z = (x, y) \in H_n$. Let $z_0 = (x, 0)$. Then $z_0 \in \partial H_n$ and $\lim\limits_{z\to z_0} f(z) = 0$ and hence for any $\varepsilon > 0$ there is $\delta > 0$

such that for $|z - z_0| < \delta, |f(z)| < \varepsilon$. Put $z_1 = (x, \frac{y}{2})$. Then

$$\frac{1}{V(K_r(z_1))} \int_{K_r(z_1)} f(w)dV(w) < \varepsilon.$$

This implies that T_f is compact.

It is enough to show that (2) implies (3) to complete the proof. Since $\lim_{z \to \infty} f(z) = 0$, for any $\varepsilon > 0$, there is $M > 0$ such that for $|z| > M$, $|f(z)| < \varepsilon$. Let $K = \{(x, y) : |x| \leq M \text{ and } 0 \leq y \leq M\}$. Then K is compact in $\mathbf{R}^n$.

Define $g : \mathbf{R}^n \longrightarrow \mathbf{C}$ by $g(z) = \begin{cases} 0 & \text{if } z_n = 0\,; \\ f(z) & \text{if } z_n \neq 0. \end{cases}$

Since $\lim_{z \to \partial H_n} f(z) = 0$, g is continuous on K and hence there is $\delta > 0$ such that for $|z_n| < \delta$, $|g(z)| < \varepsilon$. Let $K_\delta = \{z \in K : |z_n| \geq \delta\}$. Then for any $z \in H_n \setminus K_\delta$, $|f(z)| = |g(z)| < \varepsilon$. Since K_δ is compact, $f \in C_0(H_n)$. □

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