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CHARACTERIZATION OF THE HELICOID AS RULED SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

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ABSTRACT. We introduce the notion of Gauss map of pointwise 1-type on ruled surfaces in the Euclidean 3-space for which vector valued functions is neither trivial nor it extends or coincides with the usual notion of 1-type, in general. We characterize the minimal helicoid in terms of it and give a complete classification of the ruled surfaces with pointwise 1-type Gauss map.

1. Introduction

In 1970's B. Y. Chen (cf. [3, 4, 7]) defined the notion of submanifolds of finite type in Euclidean space, for which many studies have been done since then. He also extended this notion to general differential maps on submanifolds in Euclidean space E^n. For spaces other than E^n, see [10, 11, 12]. Chen and Piccini ([5]) studied the compact submanifolds in Euclidean space with Gauss map of finite type and especially characterized the compact submanifolds with 1-type and 2-type Gauss maps in Euclidean space. Other geometers have also studied submanifolds of Euclidean space or pseudo-Euclidean space with finite type Gauss map ([2, 8, 13], etc.).

On the other hand, C. Baikoussis and D. E. Blair ([1]) studied ruled surfaces in Euclidean 3-space satisfying $\Delta G = \Lambda G$, where Δ denotes the Laplace operator corresponding to the induced metric on M, G the Gauss map defined on M and Λ is a $3 \times 3-$ real matrix. In such a case, the Gauss map G can be reduced to $\Delta G = \lambda G$ for some $\lambda \in \mathbb{R}$.

However, there may be some other ruled surfaces satisfying $\Delta G = fG$ for some smooth function f. To avoid trivialities, we assume that $G \neq ha$ with h a scalar real function and a a constant vector. In such cases, we

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must again reside in the usual 1-type notion. For example, the helicoid is, up to a rigid motion, parameterized by

$$x(s,t) = (t\cos s, t\sin s, bs), \quad b \neq 0, \tag{1}$$

and the Gauss map G is given by

$$G = \frac{1}{\sqrt{t^2+b^2}}(-b\sin s, b\cos s, -t) \tag{2}$$

(see [14]). Therefore, we easily see that the Gauss map satisfies

$$\Delta G = \frac{2b^2}{(t^2+b^2)^2}G, \tag{3}$$

which is not of 1-type in usual sense.

Thus, the following question naturally arises : Besides the helicoid, which other ruled surfaces in Euclidean 3-space $\mathbb{E}^3$ satisfy

$$\Delta G = fG \tag{1.1}$$

for some real valued function f?

In this paper, we give a complete answer to this question. We now introduce the definition of pointwise 1-type Gauss map. A ruled surface M in Euclidean 3-space $\mathbb{E}^3$ is said to have *pointwise 1-type Gauss map* if it satisfies (1.1). This notion for vector valued functions F is neither trivial, if $F \neq ha$ with h scalar real function and a constant vector, nor it coincides with the usual 1-type notion, in general. Because of the trivial cases it is not a direct extension of it, either. We study the ruled surfaces in Euclidean 3-space $\mathbb{E}^3$ with pointwise 1-type Gauss map and we characterize the helicoid in terms of it.

2. Preliminaries

Let M be a surface of Euclidean 3-space $\mathbb{E}^3$. The map $G : M \to S^2(1) \subset \mathbb{E}^3$ which sends each point of M to the unit normal vector to M at the point is called the *Gauss map* of a surface M, where $S^2(1)$ denotes the unit hypersphere of $\mathbb{E}^3$.

For the matrix $g = (g_{ij})$ of the Riemannian metric on M we denote by $g^{-1} = (g^{ij})$ (resp. $\mathcal{G}$) the inverse matrix (resp. the determinant) of the matrix (g_{ij}). The Laplacian Δ on M is, in turn, given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}}\sum \frac{\partial}{\partial x^i}(\sqrt{|\mathcal{G}|}\, g^{ij}\frac{\partial}{\partial x^j}). \tag{2.1}$$

Now, we define a ruled surface M in $\mathbb{E}^3$. Let I be an open interval containing 0 in the real line $\mathbb{R}$. A ruled surface M is parametrized by

$$x(s,t) = \alpha(s) + t\beta(s), \quad s \in I, \quad t \in \mathbb{R}, \tag{2.2}$$

satisfying $\langle \alpha', \beta \rangle = 0$ and $\langle \beta, \beta \rangle = 1$. The curve $\alpha(s)$ is called a *base curve* and $\beta(s)$ a *director curve*. In particular, M is said to be *cylindrical* if $\beta(s)$ is parallel to a fixed direction in $\mathbb{E}^3$. It is called *non-cylindrical* otherwise.

We now consider the Laplacian of Gauss map of cylindrical and non-cylindrical ruled surfaces.

Case 1. M is cylindrical.

Suppose that the surface M is a cylinder over a plane curve $\alpha = (\alpha_1, \alpha_2, 0)$. We may assume that α is parameterized by its arc length s. Then, up to congruence, a parameterization x of M is given by

$$x(s,t) = \alpha(s) + t\beta, \tag{4}$$

where β is a constant vector, namely $\beta = (0,0,1)$.

The Gauss map of M is thus given by

$$G = \alpha' \times \beta, \tag{5}$$

where $\alpha' = \frac{d\alpha}{ds}$. The Laplacian Δ of M is easily obtained by

$$\Delta = -\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial t^2}. \tag{6}$$

Case 2. M is not cylindrical.

A non-cylindrical ruled surface M is parametrized, up to a rigid motion, by

$$x(s,t) = \alpha(s) + t\beta(s), \tag{7}$$

where $\alpha(s)$ is a base curve and $\beta(s)$ is a unit vector field along the rulings such that

$$\langle \alpha', \beta \rangle = 0, \quad \langle \beta, \beta \rangle = 1, \quad \langle \beta', \beta' \rangle = 1. \tag{8}$$

Then, the Gauss map G of M is given by

$$G = \frac{1}{\|(\alpha' + t\beta') \times \beta\|}((\alpha' + t\beta') \times \beta). \tag{2.3}$$

We now define three functions q, u and v by

$$q = \|\alpha' + t\beta'\|^2 = t^2 + 2ut + v, \quad u = \langle \alpha', \beta' \rangle, \quad v = \langle \alpha', \alpha' \rangle. \tag{2.4}$$

Then, the Gauss map G can be written as

$$G = q^{-\frac{1}{2}}((\alpha' + t\beta') \times \beta) = q^{-\frac{1}{2}}(A + tB), \tag{2.5}$$

where we have put $A = \alpha' \times \beta$ and $B = \beta' \times \beta$. In this case the Laplacian Δ of M can be expressed as follows (cf. [6]):

$$\Delta = -\frac{\partial^2}{\partial t^2} - \frac{1}{q}\frac{\partial^2}{\partial s^2} + \frac{1}{2}\frac{\partial q}{\partial s}\frac{1}{q^2}\frac{\partial}{\partial s} - \frac{1}{2}\frac{\partial q}{\partial t}\frac{1}{q}\frac{\partial}{\partial t}. \tag{2.6}$$

For later use, we now compute

$$\begin{aligned}
\frac{\partial G}{\partial t} &= q^{-\frac{3}{2}}\{Bq - (A + tB)(t + u)\} := q^{-\frac{3}{2}}C, \\
\frac{\partial^2 G}{\partial t^2} &= q^{-\frac{5}{2}}\{(Bu - A)q - 3(Bq - (A + tB)(t + u))(t + u)\} := q^{-\frac{5}{2}}D, \\
\frac{\partial G}{\partial s} &= \frac{1}{2}q^{-\frac{3}{2}}\{2(A' + tB')q - (A + tB)(2u't + v')\} := \frac{1}{2}q^{-\frac{3}{2}}E, \\
\frac{\partial^2 G}{\partial s^2} &= \frac{1}{2}q^{-\frac{5}{2}}\{[2(A'' + tB'')q + (A' + tB')(2u't + v') \\
&\qquad - (A + tB)(2u''t + v'')]q \\
&\qquad - \frac{3}{2}[2(A' + tB')q - (A + tB)(2u't + v')](2u't + v')\} \\
&:= \frac{1}{2}q^{-\frac{5}{2}}F,
\end{aligned}$$

where C, D, E and F are defined by the above four formulas. Using the formulas described above, the Laplacian ΔG of the Gauss map G with help of (2.6) turns out to be

$$\Delta G = -q^{-\frac{5}{2}}D - \frac{1}{2}q^{-\frac{7}{2}}F + \frac{1}{4}q^{-\frac{7}{2}}(2u't + v')E - q^{-\frac{5}{2}}(t + u)C. \tag{2.7}$$

3. A characterization of ruled surfaces

In this section we are going to characterize the ruled surfaces in terms of pointwise 1-type Gauss map, which have 1-type Gauss maps as in the usual case.

THEOREM 3.1. *The plane and the circular cylinder are the only cylindrical ruled surfaces with pointwise 1-type Gauss map.*

Proof. Let M be a cylindrical ruled surface in $\mathbb{E}^3$. The Gauss map G of M is given by

$$G = \alpha' \times \beta = (\alpha_2', -\alpha_1', 0), \tag{9}$$

and the Laplacian ΔG of the Gauss map G is derived to be $\Delta G = (-\alpha_2''', \alpha_1''', 0)$. Since M has pointwise 1-type Gauss map, (1.1) implies

$$\begin{cases} \alpha_1'''(s) = -f(s,t)\alpha_1'(s), \\ \alpha_2'''(s) = -f(s,t)\alpha_2'(s). \end{cases} \tag{3.1}$$

Since α is parametrized by the arc length, $(\alpha_1'(s))^2 + (\alpha_2'(s))^2 = 1$. Therefore, we may put

$$\alpha_1'(s) = \cos\theta(s), \quad \alpha_2'(s) = \sin\theta(s) \tag{10}$$

for some function θ of s. Putting these into (3.1), we have

$$\begin{cases} (f(s,t) - (\theta'(s))^2)\cos\theta(s) - \theta''(s)\sin\theta(s) = 0, \\ (f(s,t) - (\theta'(s))^2)\sin\theta(s) + \theta''(s)\cos\theta(s) = 0, \end{cases} \tag{11}$$

from which, we obtain

$$\theta''(s) = 0, \quad f(s,t) = (\theta'(s))^2. \tag{12}$$

Therefore, f is a constant. Thus, we conclude that M is an open portion of the plane and the circular cylinder according to Lemma of [1]. □

THEOREM 3.2. *Let M be a non-cylindrical ruled surface in $\mathbb{E}^3$. Then, M is an open portion of the helicoid if and only if the Gauss map is of pointwise 1-type.*

Proof. Suppose that M is a non-cylindrical ruled surface with pointwise 1-type Gauss map. Then, the tangential component of ΔG vanishes, that is,

$$\Delta G - \langle \Delta G, G\rangle G = 0. \tag{3.2}$$

The straightforward computation of the left hand side of (3.2) gives a polynomial in t with functions of s as the coefficients (by arranging the powers in the function q, see (2.4)) and thus they must be zero. So, we obtain

$$B'' - \langle B'', B\rangle B = 0, \tag{3.3}$$

$$\begin{aligned} A'' + (1 - \langle B'', B\rangle)A + 4uB'' - 3u'B' \\ - (u + \langle A'', B\rangle + \langle A, B''\rangle + 2u\langle B'', B\rangle)B = 0, \end{aligned} \tag{12}$$

$$
\begin{aligned}
&8uA'' - 6u'A' + 2(4u - \langle A'', B\rangle - \langle A, B''\rangle - 2u\langle B'', B\rangle)A \\
&\quad + 4(2u^2 + v)B'' - 3(4uu' + v')B' \\
&\quad - 2(3u^2 - 3u'^2 + \langle A'', A\rangle + 2u\langle A'', B\rangle + 2u\langle A, B''\rangle \\
&\quad + v\langle B'', B\rangle + v)B = 0,
\end{aligned} \tag{3.5}
$$

(3.6)
$$
\begin{aligned}
&4(2u^2 + v)A'' - 3(4uu' + v')A' \\
&\quad + 2(5u^2 + v + 3u'^2 - \langle A'', A\rangle - 2u\langle A'', B\rangle - 2u\langle A, B''\rangle - v\langle B'', B\rangle)A \\
&\quad + 8uvB'' - 6(u'v + uv')B' \\
&\quad - 2(2u^3 + 4uv - 3u'v' + 2u\langle A'', A\rangle + v\langle A'', B\rangle + v\langle A, B''\rangle)B = 0,
\end{aligned}
$$

(12)
$$
\begin{aligned}
&16uvA'' - 12(u'v + uv')A' \\
&\quad + 4(2uv + 3u'v' - 2u\langle A'', A\rangle - v\langle A'', B\rangle - v\langle A, B''\rangle + 2u^3)A \\
&\quad + 4v^2B'' - 6vv'B' - (12u^2v - 3v'^2 + 4v\langle A'', A\rangle + 4v^2)B = 0,
\end{aligned}
$$

$$
4v^2A'' - 6vv'A' + (3v'^2 - 4v\langle A'', A\rangle + 4u^2v)A - 4uv^2B = 0. \tag{3.8}
$$

From (3.3), the definition of B and the property of β we have $\langle B'', B'\rangle = 0$, and so, $\langle B', B'\rangle = c$ for some constant c. This implies that $\langle B'', B\rangle = -c$. Therefore, (3.3) gives

$$
B'' = -cB. \tag{3.9}
$$

It follows that $\langle A, B''\rangle = -cu$ since

$$
\langle A, B\rangle = \langle \alpha' \times \beta, \beta' \times \beta\rangle = \langle \alpha', \beta'\rangle = u.
$$

Consequently, the equations (3.4)-(3.8) become as follows:

$$
A'' + (1 + c)A - 3u'B' - (cu + u + \langle A'', B\rangle)B = 0, \tag{3.10}
$$

(12)
$$
\begin{aligned}
&8uA'' - 6u'A' + 2(4u - \langle A'', B\rangle + 3cu)A - 3(4uu' + v')B' \\
&\quad - 2(3u^2 - 3u'^2 + \langle A'', A\rangle + 2u\langle A'', B\rangle + 2cu^2 + cv + v)B = 0,
\end{aligned}
$$

(12)
$$
\begin{aligned}
&4(2u^2 + v)A'' - 3(4uu' + v')A' \\
&\quad + 2(5u^2 + v + 3u'^2 - \langle A'', A\rangle - 2u\langle A'', B\rangle + 2cu^2 + cv)A \\
&\quad - 6(u'v + uv')B' \\
&\quad - 2(2u^3 + 4uv - 3u'v' + 2u\langle A'', A\rangle + v\langle A'', B\rangle + 3cuv)B = 0,
\end{aligned}
$$

(12)
$$16uvA'' - 12(u'v + uv')A' + 4(2uv + 3u'v' - 2u\langle A'', A\rangle - v\langle A'', B\rangle + cuv + 2u^3)A - 6vv'B' - (12u^2v - 3v'^2 + 4v\langle A'', A\rangle + 4v^2 + 4cv^2)B = 0,$$

$$4v^2A'' - 6vv'A' + (3v'^2 - 4v\langle A'', A\rangle + 4u^2v)A - 4uv^2B = 0. \tag{3.14}$$

Using the above equations, we can eliminate A'', A' and B' so that

$$\begin{cases} (2uv'^2 - 4u'vv')A + (4u'^2v^2 - vv'^2)B = 0, \\ (-4u'^2v + v'^2)A + (8uu'^2v - 4u'vv')B = 0. \end{cases} \tag{3.15}$$

First, we suppose that A and B are linearly dependent at some $s \in I$. Then, there is a constant λ_1 such that $A(s) = \lambda_1 B(s)$, that is $\alpha' \times \beta = \lambda_1\beta' \times \beta$.
Thus, we have $\alpha' - \lambda_1\beta' = \lambda_2\beta$ for some constant λ_2. Since $\langle \alpha', \beta\rangle = 0$, we have $\lambda_2 = 0$ and thus $\alpha' = \lambda_1\beta'$. By the definition of u and v, we get

$$u = \langle \alpha', \beta'\rangle = \langle \lambda_1\beta', \beta'\rangle = \lambda_1, \tag{13}$$

$$v = \langle \alpha', \alpha'\rangle = \langle \lambda_1\beta', \lambda_1\beta'\rangle = {\lambda_1}^2. \tag{14}$$

So we have $u^2 = v$. This contradicts the property of the smooth positive function $q$in (2.4). Thus, A and B are linearly independent for all s. Thereby, from (3.15), we obtain the following equations:

$$uv'^2 - 2u'vv' = 0, \tag{3.16}$$

$$v'^2 = 4u'^2v, \tag{3.17}$$

$$2uu'^2v - u'vv' = 0. \tag{3.18}$$

If $u' \neq 0$, we obtain $v' = 2uu'$ by virtue of (3.18). In this case, we get $u^2 = v$ from (3.17). This is also a contradiction. Thus, we have

$$u' = 0. \tag{3.19}$$

This together with (3.17) yields

$$v' = 0. \tag{3.20}$$

On the other hand, (3.9) implies

$$\beta''' \times \beta + \beta'' \times \beta' + c\beta' \times \beta = 0. \tag{3.21}$$

Taking the inner product with β in (3.21), we obtain

$$\langle \beta'' \times \beta', \beta\rangle = 0. \tag{15}$$

Since in this case $\beta \neq 0$, $\beta' \neq 0$, $\beta'' \neq 0$, there are smooth functions μ_1 and μ_2 such that

$$\beta = \mu_1 \beta'' + \mu_2 \beta'. \tag{16}$$

Since $\langle \beta', \beta \rangle = 0$, we have $\mu_2 = 0$ and so $\beta = \mu_1 \beta''$ (and $\mu_1 \neq 0$). But, by the definition we have u and v that

$$u' = \langle \alpha'', \beta' \rangle + \langle \alpha', \beta'' \rangle, \quad v' = 2\langle \alpha'', \alpha' \rangle. \tag{17}$$

Since $\langle \alpha', \beta \rangle = 0$ and $\beta = \mu_1 \beta''$, we have $u' = \langle \alpha'', \beta' \rangle$. Consequently, (3.19) and (3.20) yield

$$\langle \alpha'', \beta' \rangle = 0, \quad \langle \alpha'', \alpha' \rangle = 0. \tag{3.22}$$

For the vector fields α', β, β' and α'' we may put

$$\alpha'' = k_1 \alpha' + k_2 \beta' + k_3 \beta, \tag{18}$$

where k_1, k_2 and k_3 are some smooth functions. The last equation together with (3.22) gives $k_1 = 0$ and $k_2 = 0$, in other words, α'' and β are parallel.

On the other hand, by definition the mean curvature vector field H of M is easily obtained as follows :

$$H = \frac{1}{2} q^{-\frac{3}{2}} \langle (\alpha' + t\beta') \times \beta, \alpha'' + t\beta'' \rangle. \tag{19}$$

Since β'', α'' and β are parallel to each other, it is easily proved that H vanishes identically. Thus, M is an open part of the helicoid (The helicoid is the only minimal ruled surfaces, except for the plane, see [9], p. 204). The converse is obvious and this completes the proof. □

Combining the results of Theorem 3.1 and 3.2, we have the following

THEOREM 3.3. (Classification) *The ruled surfaces in* $\mathbb{E}^3$ *with pointwise 1-type Gauss map are the open portions of the plane, the circular cylinder and the minimal helicoid.*

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