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LOCALIZATION OF THE COHOMOLOGY OF THE HOMOTOPY ORBIT SPACE OF p-COMPACT GROUPS

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ABSTRACT. For a p-compact group G, if G-space X is of finite S-type then we show that the localization $S^{-1}H^*(X_{hG})$ is zero. By using this result, we prove the localization theorem for the pair G-space (X, A).

1. Introduction

A loop space is a triple $G = (G, BG, e)$, where G is a topological space, BG is a connected pointed classifying space of G and $e : G \to \Omega BG$ is a homotopy equivalence from G to the space ΩBG of based loops in BG. Such a loop space is called *p-compact group* if G is $\mathbb{F}_p$-finite and BG is $\mathbb{F}_p$-complete. Here the second condition is equivalent to that G is $\mathbb{F}_p$-complete and $\pi_0(G)$ is a finite p-group. The main example of p-compact group is the p-completion of compact Lie group G, $(\widehat{G_p}, \widehat{BG_p}, e)$, where $\pi_0(G)$ is a finite p-group and $e : \Omega \widehat{BG_p} \simeq \widehat{G_p}$. Dwyer and Wilkerson defined these p-compact groups and proved a lot of their properties in [5]. Their work shows that a p-compact group has much of the rich internal structure of a compact Lie group. In particular, they showed that every p-compact group has a maximal torus, normalizer of the maximal torus and Weyl groups. More homotopy theories of p-compact groups are developed in [6], [7], and [8].

Let G be a p-compact group. A *G-space* X is defined to be a fibration $p_X : X_{hG} \to BG$ with X as the fibre. Here we say X_{hG} to be *homotopy orbit space* of a p-compact group G. In this paper we give localization properties of $H^*(X_{hG})$ for p-compact group G. This is the generalization of the localization theorem for the equivariant cohomology of compact

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Lie group. In Section 2 we give basic definitions and properties regarding p-compact groups as preliminaries. Section 3 gives new definitions with respect to the p-compact groups and the proof of our main results.

All unspecified cohomology $H^*(\_)$ groups are assumed with coefficients in $\mathbb{F}_p$.

2. Preliminaries

A graded vector space H^* over a field F is *finite dimensional* if each H^i is finite dimensional over F and $H^i = 0$ for all but finite number of i. A space X is $\mathbb{F}_p$*-finite* if H^*X is finite dimensional over a finite field $\mathbb{F}_p$. Let $\epsilon_X : X \to X_p\hat{}$ be a natural map for any space X where $(\_)_p\hat{}$ is $\mathbb{F}_p$-completion functor constructed by Bousfield and Kan [2]. If ϵ_X is homotopy equivalent, we say X is $\mathbb{F}_p$*-complete.*

Now we give the basic definitions regarding the p-compact groups [5].

A *homomorphism* $f : K \to G$ of p-compact groups is a pointed map $Bf : BK \to BG$. The homogeneous space G/K is defined to be the homotopy fiber of Bf over the basepoint of BG. The homomorphism f is said to be *monomorphism* or equivalently $K \to G$ is a *subgroup* of G if the homotopy fibre G/K of Bf is $\mathbb{F}_p$-finite, and an *epimorphism* if $\Omega(G/K)$ is a p-compact group. Two homomorphism $f_1, f_2 : K \to G$ are *conjugate* if the associated maps $Bf_1, Bf_2 : BK \to BG$ are freely homotopic. A *short exact sequence* $K \xrightarrow{f} H \xrightarrow{g} G$ of p-compact groups is sequence such that $BK \xrightarrow{Bf} BH \xrightarrow{Bg} BG$ is a fibration sequence where f is a monomorphism and g is an epimorphism.

Let G be a p-compact group. We define G*-space* X to be the fibration $p_X : X_{hG} \to BG$ with X as the fibre. The G*-equivariant map* $X \to Y$ is defined to be a map of spaces together with an extension to a map $X_{hG} \to Y_{hG}$ of spaces over BG. A G*-subspace* A of X is a subspace A with the fibration $A_{hG} \to BG$ with a homotopy fibre A. If A is a G-subspace of X, then A_{hG} is a subspace of X_{hG}. For $i = 1, 2$, let $f_i : H_i \to G$ be subgroups of G. Then H_1 is *subconjugate* to H_2 if there exists a homomorphism $h : H_1 \to H_2$ such that $f_2 \circ h$ and f_1 are conjugate. We say that the p-compact subgroups $H_1 \to G$ and $H_2 \to G$ are *conjugate* in G, denoted by $H_1 \sim H_2$, if H_1 and H_2 are subconjugate to each other.

In the next section we extend the localization theorem for equivariant cohomology of compact Lie group to a theorem for p-compact group.

3. Localization of $H^*(X_{hG})$ for p-compact group G

Let G be a p-compact group. We give the localization property of the cohomology of the homotopy orbit space of a p-compact group G.

The following result is known by W. G. Dwyer and C. W. Wilkerson.

THEOREM 3.1 ([5]). *If G is a p-compact group, then $H^*(BG, \mathbb{F}_p)$ is finitely generated as an algebra.*

From classical algebra, if X is connected then H^*X is finitely generated as an algebra if and only if H^*X is Noetherian as a graded ring if and only if every graded ideal in H^*X has a finite number of homogeneous generators if and only if every graded submodule of a graded finitely generated H^*X-module is itself finitely generated. Also a H^*X-module satisfies the ascending chain condition on submodules if and only if every submodule of H^*X-module is finitely generated.

Now we give the following definitions.

DEFINITION. An *isotropy family* for the p-compact group G is a set $\mathcal{F}$ of p-compact subgroups $H \to G$ such that if $H \to G$ belongs to $\mathcal{F}$ and $H \sim K$, then $K \to G$ also belongs to $\mathcal{F}$. The isotropy family $\mathcal{F}$ is said to be *open* if $H \to G$ belongs to $\mathcal{F}$ and $K \to H$, a subgroup of H, implies that $K \to G$ belongs to $\mathcal{F}$. The isotropy family $\mathcal{F}$ is said to be *closed* if $K \to G$ belongs to $\mathcal{F}$ and $K \to H$, a subgroup of H, implies that $H \to G$ belongs to $\mathcal{F}$.

DEFINITION. A G-space X is *$\mathcal{F}$-numerable* if there exists a covering $\mathcal{U} = \{U_i \mid i \in I\}$ of X by G-subspaces with the following properties.

(i) For each $i \in I$, there exists a G-equivariant map

$$f_i : U_i \to G/G_i, \quad \text{where } G_i \to G \text{ belongs to } \mathcal{F}.$$

(ii) There exists a locally finite partition of unity $(t_i \mid i \in I)$ subordinate to $\mathcal{U}$ by G-functions $t_i : X \to [0, 1]$.

If $f : X \to Y$ is G-equivariant and Y is $\mathcal{F}$-numerable, then X is also $\mathcal{F}$-numerable. For this, we take open G-covering $\mathcal{U} = \{U_i \mid U_i = f^{-1}(V_i), V_i \in \mathcal{V}\}$ where $\mathcal{V}$ is a G-covering of Y satisfying the $\mathcal{F}$-numerable condition. Then the composition $U_i \to V_i \to G/G_i$ is also G-equivariant for $G_i \to G$ in $\mathcal{F}$. If we set $t_i = s_i \circ f$ where $\{s_i \mid s_i : Y \to [0,1], i \in I\}$ is a locally finite partition of unity subordinate to $\mathcal{V}$, then it is easy to see that $\{t_i \mid i \in I\}$ is also a locally finite partition of unity subordinate to $\mathcal{U}$.

Let X be a G-space with a fibration $X \xrightarrow{i} X_{hG} \xrightarrow{p_X} BG$.

We assume $H^*(\_)$ has its usual multiplicative structure. This means that we are given product pairings (cup product)

$$H^m(X,A) \otimes H^n(X,B) \to H^{m+n}(X, A \cup B)$$

with the usual properties. This product yields a product pairing

$$H^m(X_{hG}, A_{hG}) \otimes H^n(X_{hG}, B_{hG}) \to H^{m+n}(X_{hG}, A_{hG} \cup B_{hG}).$$

Now $H^*(X_{hG})$ is a graded module over $H^*(BG)$ in a canonical way. The module structure is defined as follows. For $b \in H^*(BG)$ and $x \in H^*(X_{hG})$, consider $p_X^*(b) \in H^*(X_{hG})$ and form the product $p_X^*(b) \cup x$. In particular $H^*(X_{hG})$ becomes a graded algebra with unit.

Let $S \subset H^*(BG)$ be a multiplicatively closed subset of homogeneous elements. We assume S is contained in the center of $H^*(BG)$. Let M be a graded $H^*(BG)$-module and consider the localization of M with respect to S, denoted by $S^{-1}M$. Regard $S^{-1}M$ as $S^{-1}H^*(BG)$-module and

$$S^{-1}M \cong S^{-1}H^*(BG) \otimes_{H^*(BG)} M.$$

The following result is from the commutative algebra.

PROPOSITION 3.2 ([1]).

(i) *If $M \to N \to P$ is an exact sequence of $H^*(BG)$-modules, then $S^{-1}M \to S^{-1}N \to S^{-1}P$ is an exact sequence of $S^{-1}H^*(BG)$-modules.*

(ii) *The kernel of the canonical map $\phi : M \to S^{-1}M$ consists of those $m \in M$ which are annihilated by some element of S.*

(iii) *Localization commutes with colimits.*

We consider the following set of p-compact subgroups of G.

$$\mathcal{F}(S) = \{H \to G \mid S \cap \mathit{kernel}(H^*(BG) \xrightarrow{p_H^*} H^*((G/H)_{hG})) \neq \emptyset\}$$

where p_H^* is the induced homomorphism on the cohomology from the fibration $(G/H)_{hG} \xrightarrow{p_H} BG$.

PROPOSITION 3.3. *The set of p-compact subgroups of G, $\mathcal{F}(S)$ is an isotropy family.*

Proof. We need to show that if $H \to G$ belongs to $\mathcal{F}$ and $H \sim K$, then $K \to G$ belongs to $\mathcal{F}$. We consider the fibrations $p_H : (G/H)_{hG} \to BG$ and $p_K : (G/K)_{hG} \to BG$. Since $H \sim K$, $(G/H)_{hG} \simeq BH$ and $(G/K)_{hG} \simeq BK$, there exist $\bar{f} : (G/H)_{hG} \to (G/K)_{hG}$ and $\bar{g}$:

$(G/K)_{hG} \to (G/H)_{hG}$ such that $p_K \circ \bar{f} \simeq p_H$ and $p_H \circ \bar{g} \simeq p_K$. Then $\bar{f}^* \circ p_K^* = p_H^*$ and $\bar{g}^* \circ p_H^* = p_K^*$. Let x belong to $S \cap \ker p_H^*$. For such $x \in S, p_H^*(x) = 0$ and $p_K^*(x) = \bar{g}^* \circ p_H^*(x) = 0$. Thus $x \in \ker p_K^*$, and hence $x \in S \cap \ker p_K^*$. This implies $S \cap \ker p_K^* \neq \emptyset$. Therefore $K \to G$ belongs to $\mathcal{F}$. □

DEFINITION. A G-space X is of *finite S-type* if there exist a numerable, finite dimensional G-covering $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ of X, a finite number of p-compact subgroups $H_1 \to G, \cdots, H_r \to G$ in $\mathcal{F}(S)$ and G-equivariant maps $f_\alpha : U_\alpha \to G/H_{n(\alpha)}, n(\alpha) \in \{1, 2, \cdots, r\}$.

It is easily checked that if X is of finite S-type and $Y \subset X$, then Y is of finite S-type.

LEMMA 3.4. *Suppose $U_1, U_2, \cdots, U_r$ is an open G-covering of X. If we are given elements $x_i \in H^*(X_{hG})$ whose restriction to U_i is zero, then the product $x_1 \cdots x_r$ is zero.*

Proof. Since U_i is a G-subspace of X, $(U_i)_{hG}$ is a subspace of X_{hG} for $i = 1, 2, \cdots, r$. For each pair space $(X_{hG}, (U_i)_{hG})$, there exists a long exact sequence

$$\cdots \to H^n(X_{hG}, (U_i)_{hG}) \xrightarrow{j^*} H^n(X_{hG}) \xrightarrow{i^*} H^n((U_i)_{hG}) \xrightarrow{\delta_*} H^{n+1}(X_{hG}, (U_i)_{hG}) \to \cdots .$$

Given element $x_i \in H^*(X_{hG})$, there exists $y_i \in H^n(X_{hG}, (U_i)_{hG})$ such that $i^* \circ j^*(y_i) = i^*(x_i) = 0$ by exactness. Then the product $y_1 \cdots y_r$ is defined and contained in $H^*(X_{hG}, (U_1)_{hG} \cup \cdots \cup (U_r)_{hG})$. However

$$H^*(X_{hG}, (U_1)_{hG} \cup \cdots \cup (U_r)_{hG}) = 0.$$

Therefore $y_1 \cdots y_r$ is zero. This implies $x_1 \cdots x_r$ is zero. □

THEOREM 3.5. *Let X be of finite S-type. Then $S^{-1}H^*(X_{hG})$ is zero.*

Proof. Since X is of finite S-type, there exist finite number of p-compact subgroups $H_1 \to G, H_2 \to G, \cdots, H_r \to G$ in $\mathcal{F}(S)$ and G-equivariant maps $f_\alpha : U_\alpha \to G/H_{n(\alpha)}$, $n(\alpha) \in \{1, 2, \cdots, r\}$. Let $A_i = \cup\{U_\alpha \mid n(\alpha) = i\}$. Then $A_1, A_2, \cdots, A_r$ is open G-covering of X. If we show that for each i there exists $s_i \in S$ with image in $H^*((A_i)_{hG})$ being zero, then the product $s = s_1 s_2 \cdots s_r$ is zero in $H^*(X_{hG})$ by Lemma 3.4. Hence each element in $H^*(X_{hG})$ is annihilated by s. This implies $\ker\phi$ is equal to $H^*(X_{hG})$ for the canonical map $\phi : H^*(X_{hG}) \to S^{-1}H^*(X_{hG})$, and hence $S^{-1}H^*(X_{hG})$ is zero. Since the covering $\{U_\alpha \| \, n(\alpha) = i\}$ of

A_i is finite dimensional and numerable, it is sufficient to consider the case $r = 1$. Let $H = H_1$, $A = \cup\{U_\alpha \mid n(\alpha) = 1\}$ and there exists $H \to G \in \mathcal{F}(S)$ with G-equivariant maps $f_\alpha : U_\alpha \to G/H$. Then there exists a covering $V_0, \cdots, V_n$ of X such that each V_i is a disjoint union of open G-sets which are contained in at least one of U_α. In particular, each V_i has a G-equivariant map $h_i : V_i \to G/H$. Since $H \to G$ belongs to $\mathcal{F}(S)$, there exists $s \in S$ in the kernel of $p_H^* : H^*(BG) \to H^*((G/H)_{hG})$. Then s belongs to the kernel of the composition

$$H^*(BG) \to H^*((G/H)_{hG}) \to H^*((V_i)_{hG})$$

where the second map is $(h_i)_{hG}^*$. Thus $H^*((V_i)_{hG})$ is annihilated by s. Therefore $H^*(X_{hG})$ is annihilated by s^{n+1}. This completes the proof. □

DEFINITION. The G-subspace A of X is *taut* in X with respect to $H^*(\_)$ if the canonical map

$$\operatorname{colim}_V H^*(X_{hG}, V_{hG}) \to H^*(X_{hG}, A_{hG})$$

is an isomorphism where the colimit is taken over the open G-neighborhoods V of A in X.

THEOREM 3.6. *Let A be taut in X and closed. Let $X \backslash A$ be of finite S-type. Then the inclusion map $A \hookrightarrow X$ induces an isomorphism*

$$S^{-1}H^*(X_{hG}) \approx S^{-1}H^*(A_{hG}).$$

Proof. Since localization preserves exactness, it is sufficient to show that $S^{-1}H^*(X_{hG}, A_{hG})$ is zero. Since localization commutes with colimits, it suffices to show that $S^{-1}H^*(X_{hG}, V_{hG})$ is zero for open G-neighborhoods V of A. Since A is closed in X, we have the following excision isomorphism

$$S^{-1}H^*(X_{hG}, V_{hG}) \approx S^{-1}H^*((X\backslash A)_{hG}, (V\backslash A)_{hG}).$$

However $X\backslash A$ is of finite S-type, and hence $V\backslash A$ is of finite S-type. Thus $S^{-1}H^*((X\backslash A)_{hG})$ and $S^{-1}H^*((V\backslash A)_{hG})$ are zeros by Theorem 3.5.

By exact cohomology sequence for the pair space $((X\backslash A)_{hG}, (V\backslash A)_{hG})$, $S^{-1}H^*((X\backslash A)_{hG}, (V\backslash A)_{hG})$ is zero. Therefore $S^{-1}H^*(X_{hG}, V_{hG})$ is zero. This completes the proof. □

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